TECHNISCHE UNIVERSITÄT BERLIN

Geometry and combinatorics of max-linear Bayesian networks

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What is a MLBN?

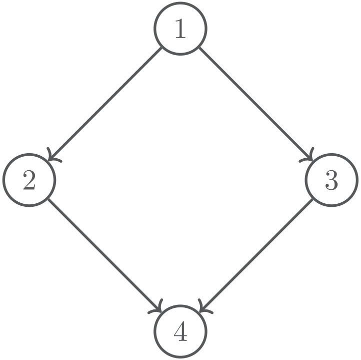
Definition

A max-linear Bayesian network is a random vector $X = (X_1, \dots, X_n)$ where the random variables X_i satisfy the recursive equations

$$X_i = \bigvee_{j=1}^{n} c_{ij} X_j \vee Z_i$$

for some $C \in \mathbb{T}^{n \times n}$.

The first contact



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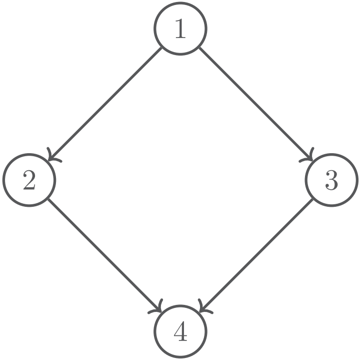

$$X_1 = Z_1$$
$$X_2 = c_{12}X_1 \vee Z_2$$
$$X_3 = c_{13}X_1 \vee Z_3$$
$$X_4 = c_{24}X_2 \vee c_{34}X_3 \vee Z_4$$

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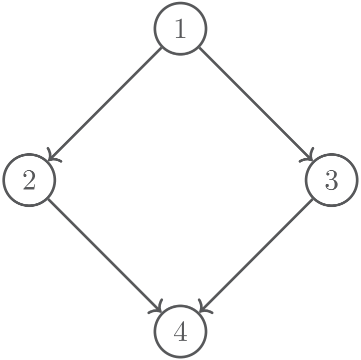

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_{12} & 0 & 0 & 0 \\ c_{13} & 0 & 0 & 0 \\ 0 & c_{24} & c_{34} & 0 \end{pmatrix} \cdot X \vee Z$$

Outline

- What is a max-linear Bayesian network?
- Tropical polytopes and regular subdivisions
- MLBNs and shortest paths
- Conditional independence

Tropical polytopes and regular subdivisions

Tropical semirings

Definition

The *min-plus tropical semiring* is $\mathbb{T} := \mathbb{T}_{\min} := (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ where $\oplus := \min$ and $\odot := +$.

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Alternatively, $\mathbb{T}_{\max} := (\mathbb{R} \cup \{-\infty\}, \max, +)$, $(\mathbb{R}_{\geq 0}, \min, \cdot)$, and $(\mathbb{R}_{\geq 0}, \max, \cdot)$ are equally valid choices for tropical semirings.

$$
\begin{array}{ccc}
\mathbb{T}_{\min} & \xleftrightarrow{\cdot(-1)} & \mathbb{T}_{\max} \\
\log \uparrow \; \downarrow exp & & \log \uparrow \; \downarrow exp \\
(\mathbb{R}_{\geq 0}, \min) & \xleftrightarrow{\frac{1}{\cdot}} & (\mathbb{R}_{\geq 0}, \max)
\end{array}
$$

Tropical affine space

Definition

Tropical affine space refers to $\mathbb{TA}^{d-1} := \mathbb{R}^d/\mathbb{R}\mathbb{1}$, that is, the Euclidean space modulo the equivalence relation

$$(x_1, \ldots, x_d) \equiv (x_1 + \lambda, \ldots, x_d + \lambda), \quad \lambda \in \mathbb{R}.$$

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For the tropical semirings $\mathbb{T}_{\min}$ and $\mathbb{T}_{\max}$ we can identify $\mathbb{TA}^{d-1}$ with Euclidean space $\mathbb{R}^{d-1}$ via

$$(x_1, \dots, x_n) \equiv (0, x_2 - x_1, \dots, x_n - x_1).$$

Tropical polytopes

Definition

The *tropical convex hull* of $V \in \mathbb{T}^{d\times n}$ finite is the *(min-plus)-linear span* of V, i. e.

$$\operatorname{tconv}(V) := \left\{ \lambda_1 \odot v^{(1)} \oplus \cdots \oplus \lambda_n \odot v^{(n)} \;\middle|\; \lambda_i \in \mathbb{R} \right\}.$$

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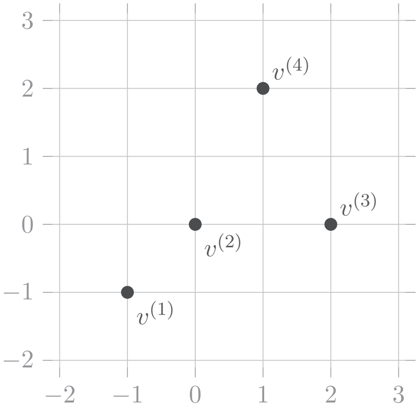
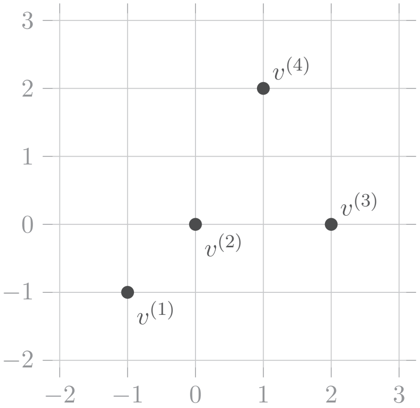
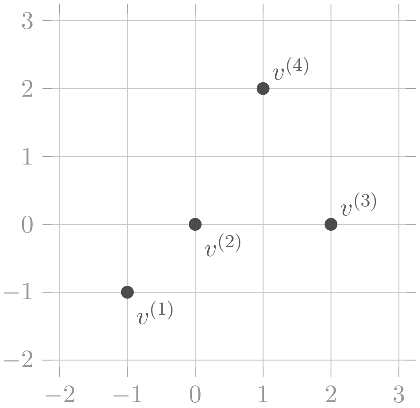


$$V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$



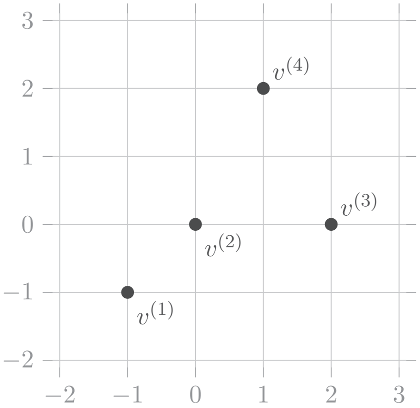
$$V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

$$\operatorname{tconv}(v^{(1)}, v^{(3)}) \ni \underbrace{\begin{pmatrix} \lambda_1 & \oplus & \lambda_3 \\ \lambda_1 - 1 & \oplus & \lambda_3 + 2 \\ \lambda_1 - 1 & \oplus & \lambda_3 \end{pmatrix}}_{=:v}$$



Assume $\lambda_3 < \lambda_1$:

$$v = \begin{pmatrix} & \lambda_3 & \\ \lambda_1 - 1 & \oplus & \lambda_3 + 2 \\ \lambda_1 - 1 & \oplus & \lambda_3 \end{pmatrix}$$



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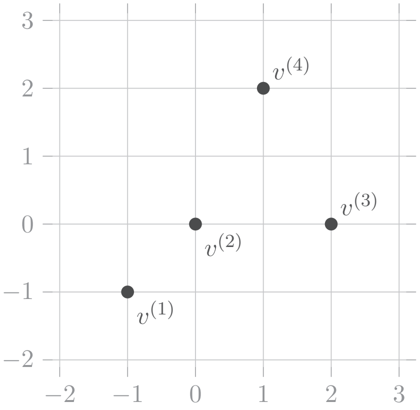

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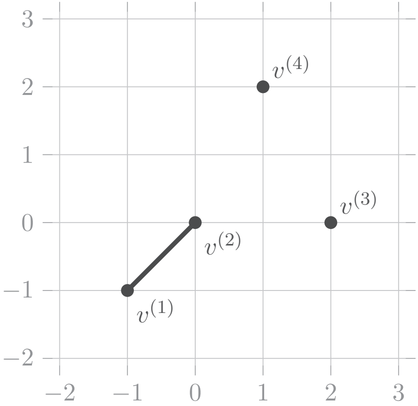
where $\lambda \in [0, 1]$.



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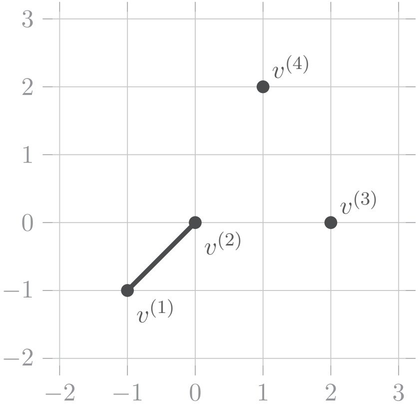

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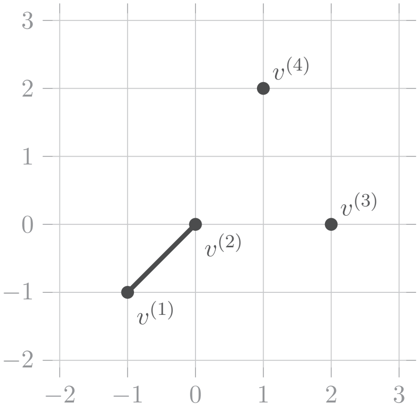

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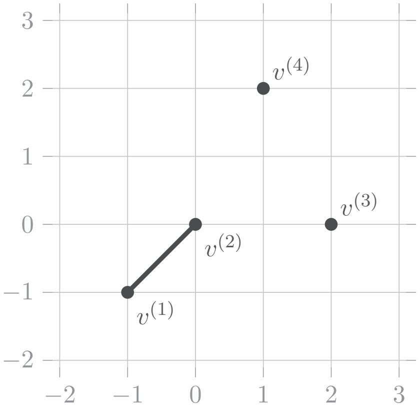

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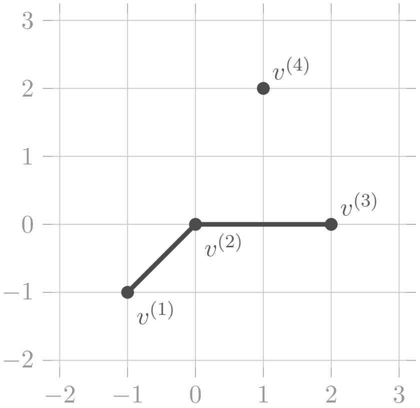

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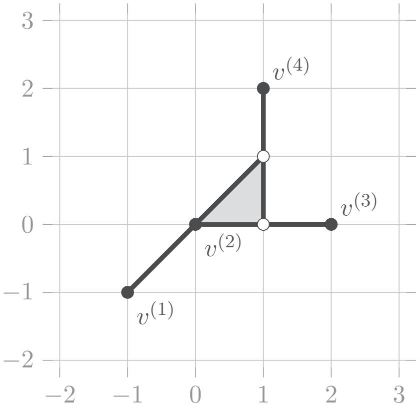

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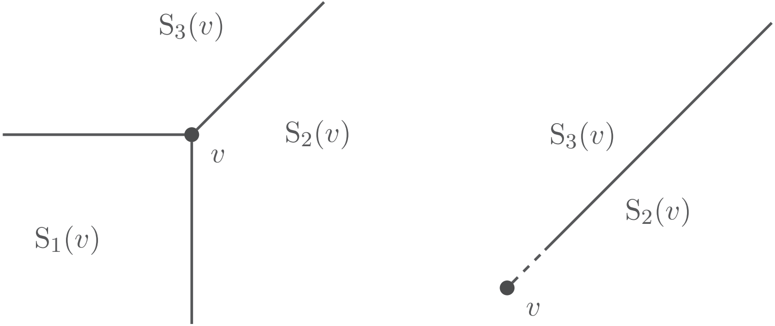
Tropical hyperplanes



$$\alpha_v = -v_1 \odot x_1 \boxplus -v_2 \odot x_2 \boxplus -v_3 \odot x_3$$

Tropical hyperplane arrangements

Definition

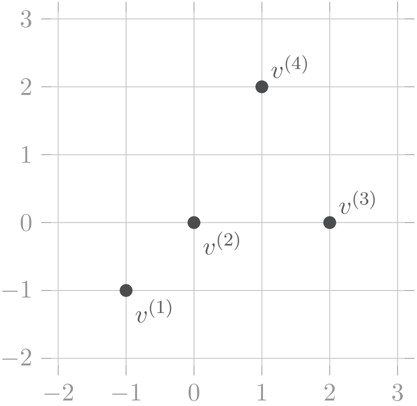
The *tropical hyperplane arrangement* $\mathcal{T}(V)$ is the tropical hypersurface defined by the polynomial

$$\mathcal{Q}_V := \prod_{j=1}^{n} \alpha_{v(j)} \in \mathbb{T}_{max}[x_1, \ldots, x_d].$$

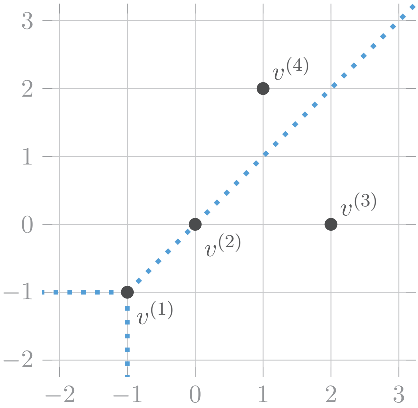
That is, $\mathcal{T}(V)$ is the set of points $x \in \mathbb{TA}^{d-1}$ such that the minimum in $\mathcal{Q}_V$ is achieved by at least two terms.

Theorem (Fink and Rincón 2015)

The arrangement $\mathcal{T}(V)$ induces a polyhedral subdivision of $\mathbb{TA}^{d-1}$ called the tropical hyperplane complex of V. Furthermore, $\mathrm{tconv}(V)$ is a subcomplex of this tropical hyperplane complex.



$$
\begin{aligned}
\mathcal{Q}_V = & \; (x_1 \boxplus x_2 + 1 \boxplus x_3 + 1) \\
& + (x_1 \boxplus x_2 \boxplus x_3) \\
& + (x_1 \boxplus x_2 - 2 \boxplus x_3) \\
& + (x_1 \boxplus x_2 - 1 \boxplus x_3 - 2)
\end{aligned}
$$



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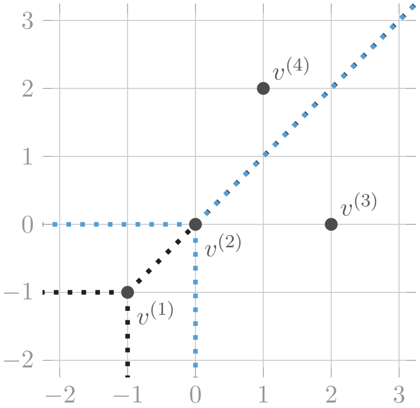

$$
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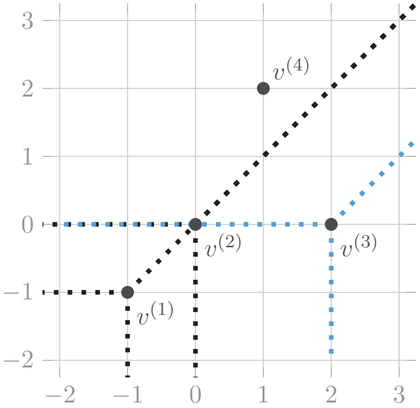
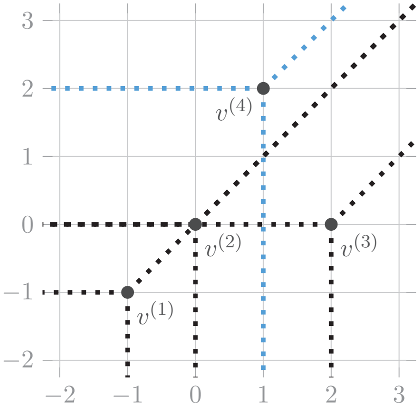
\end{aligned}
$$



$$\begin{aligned}\mathcal{Q}_V = &(x_1 \boxplus x_2 + 1 \boxplus x_3 + 1)\\ &+ (x_1 \boxplus x_2 \boxplus x_3)\\ &+ (x_1 \boxplus x_2 - 2 \boxplus x_3)\\ &+ (x_1 \boxplus x_2 - 1 \boxplus x_3 - 2)\end{aligned}$$



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+ {} & (\; x_1 \boxplus x_2 \boxplus x_3) \\
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\end{aligned}
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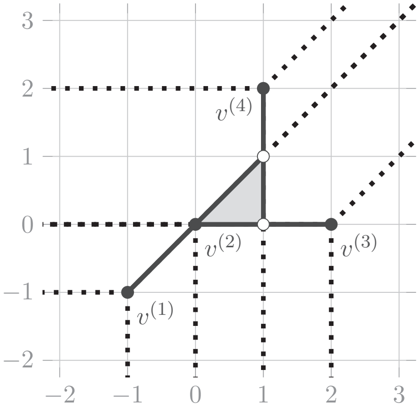

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Regular subdivisions

Fix a finite set $A \subset \mathbb{R}^n$ of m points.

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Lift every point in A to some height $h(a) \in \mathbb{R}$ and take the convex hull of this in $\mathbb{R}^{n+1}$. Project the lower faces back to $\mathbb{R}^n$. This gives a *regular subdivision* of A.

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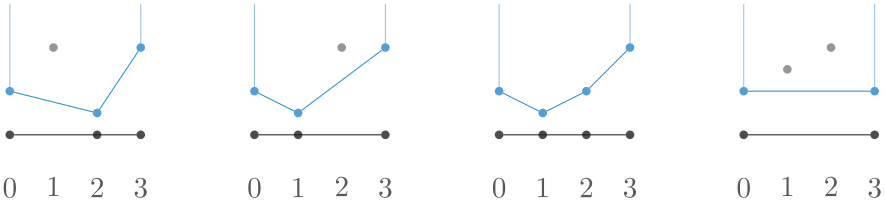

Newton polyhedra

Denote by $\operatorname{supp}(F)$ the support of a tropical polynomial $F \in \mathbb{T}[x_1, \dots, x_d]$, i.e. the set of exponent vectors $s \in \mathbb{Z}^d$ such that the coefficient $c_s \neq \infty$.

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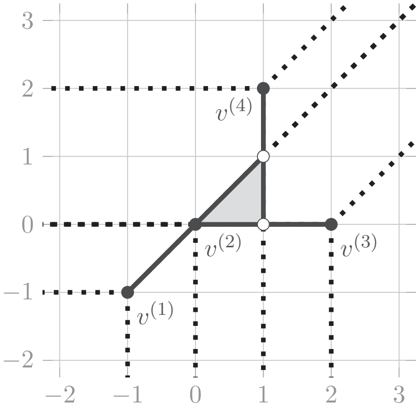
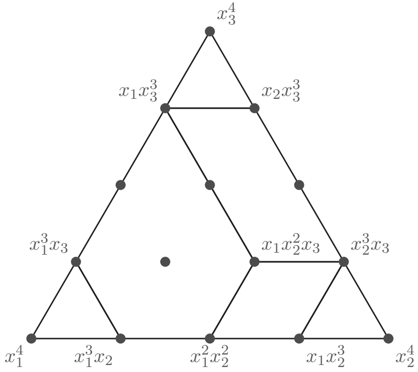
Theorem (Fink and Rincón 2015)

The tropical hyperplane complex of V is dual to the mixed regular subdivision of $\mathcal{N}(\mathcal{Q}_V) \subseteq n \cdot \Delta_{d-1}$ with height function given by the coefficients of $\mathcal{Q}_V$.


3
2
1
0
−1
−2
−2
−1
0
1
2
3
$v^{(4)}$
$v^{(2)}$
$v^{(3)}$
$v^{(1)}$


x_3^4
$x_1x_3^3$
$x_2x_3^3$
$x_1^3x_3$
$x_1x_2^2x_3$
$x_2^3x_3$
x_1^4
$x_1^3x_2$
$x_1^2x_2^2$
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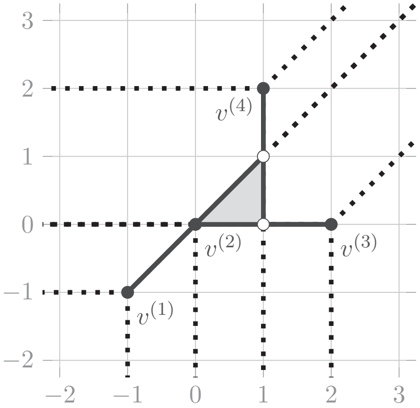
x_2^4



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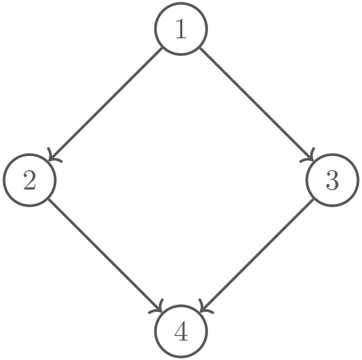
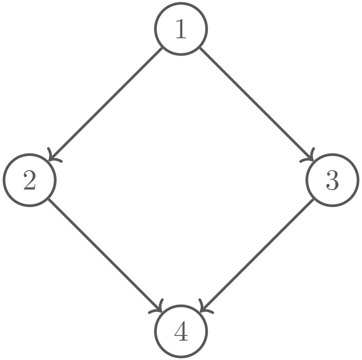
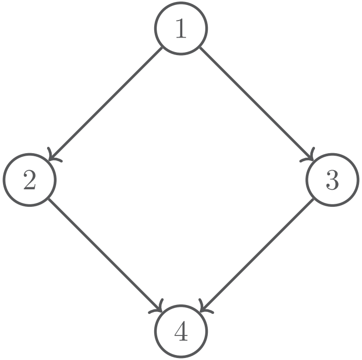
\end{aligned}
$$

MLBNs and shortest paths


1
2
3
4



$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_{12} & 0 & 0 & 0 \\ c_{13} & 0 & 0 & 0 \\ 0 & c_{24} & c_{34} & 0 \end{pmatrix} \cdot X \vee Z$$



$$X = \begin{pmatrix} \infty & \infty & \infty & \infty \\ c_{12} & \infty & \infty & \infty \\ c_{13} & \infty & \infty & \infty \\ \infty & c_{24} & c_{34} & \infty \end{pmatrix} \odot X \oplus Z$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} \infty & \infty & \infty & \infty \\ c_{12} & \infty & \infty & \infty \\ c_{13} & \infty & \infty & \infty \\ \infty & c_{24} & c_{34} & \infty \end{pmatrix} \odot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \oplus \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}$$

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Tropical polytopes and polytropes

Definition

The *tropical convex hull* of $V \in \mathbb{T}^{d\times n}$ finite is the *(min-plus)-linear span* of V, i. e.

$$\operatorname{tconv}(V) := \left\{ \lambda_1 \odot v^{(1)} \oplus \cdots \oplus \lambda_n \odot v^{(n)} \;\middle|\; \lambda_i \in \mathbb{R} \right\}.$$

Note: A tropical polytope P is the column span of a matrix with the vertices of P as columns.

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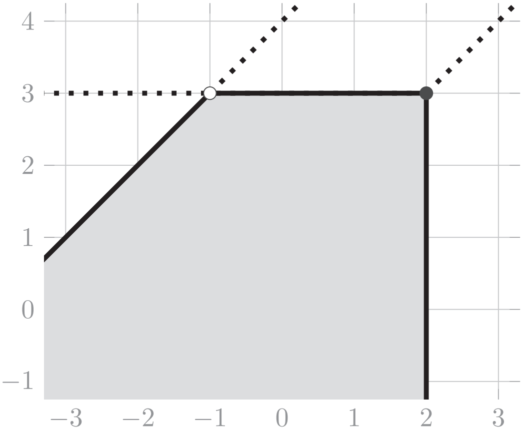
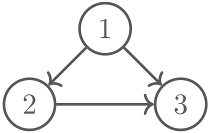
Definition (Joswig and Kulas 2010)

A tropical polytope P is called a polytrope if it is classically convex.





$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix}$$

Polyhedra from weighted digraphs

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$$x_i - x_j \leq c_{ij}.$$

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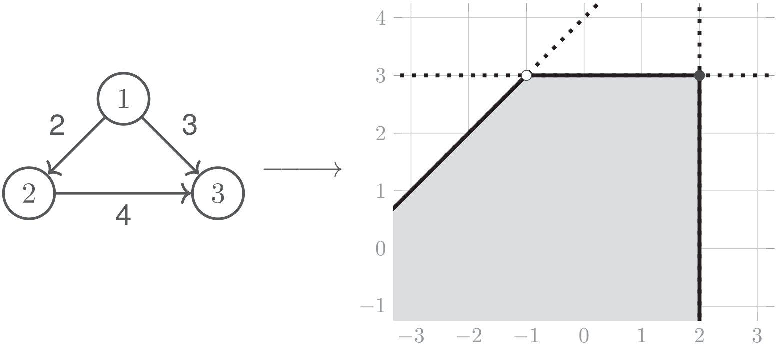

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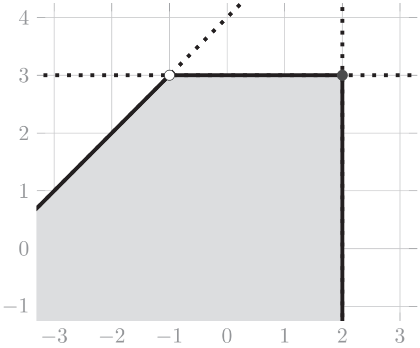
$$\begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix} \longrightarrow$$



Polyhedra from weighted digraphs

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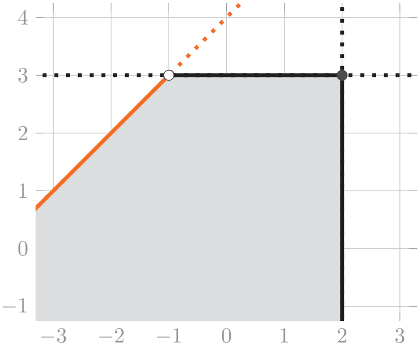

Tropical polytopes and polytropes

Definition (Joswig and Kulas 2010)

A tropical polytope P is called a polytrope if it is classically convex.

Theorem (Butkovič 2010; Joswig and Loho 2016; Puente 2013)

A tropical polytope $P \subset \mathbb{TA}^{d-1}$ is a polytrope if and only if

$$P = \operatorname{tconv}(C^{\star}) = \mathrm{Q}(C) = \mathrm{Q}(C^{*}) \subseteq \operatorname{tconv}(C)$$

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Note: $C^\star = I_n \oplus A \oplus A^2 \oplus \cdots \oplus A^n$ is the *Kleene star* of C.

Regular dual subdivisions and tropical polytopes

Theorem (Fink and Rincón 2015)

The arrangement $\mathcal{T}(V)$ induces a polyhedral subdivision of $\mathbb{TA}^{d-1}$ called the hyperplane complex of V. $\operatorname{tconv}(V)$ is a subcomplex of the hyperplane complex.

Theorem (Fink and Rincón 2015)

The tropical hyperplane complex of V is dual to the mixed regular subdivision of $\mathcal{N}(\mathcal{Q}_V) \subseteq n \cdot \Delta_{d-1}$ with height function given by the coefficients of $\mathcal{Q}_V$.

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Theorem

The cell poset of $\mathrm{tconv}(C)$ is dual to the regular subdivision of $\mathcal{N}(\mathcal{Q}_V)$ induced by C intersected with the interior of $n\Delta_{d-1}$.

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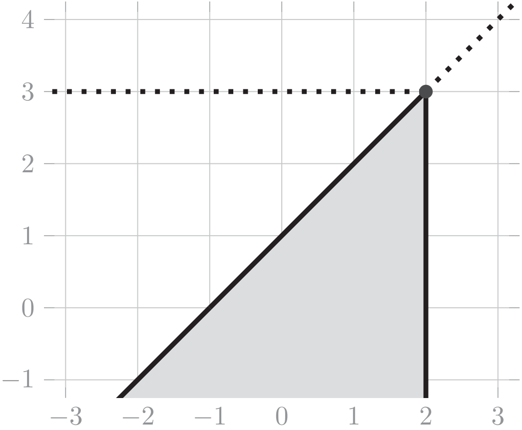
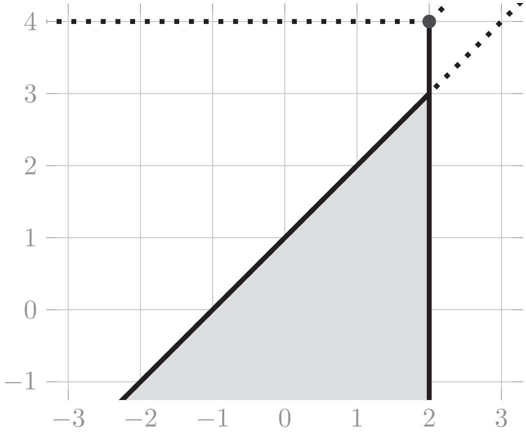


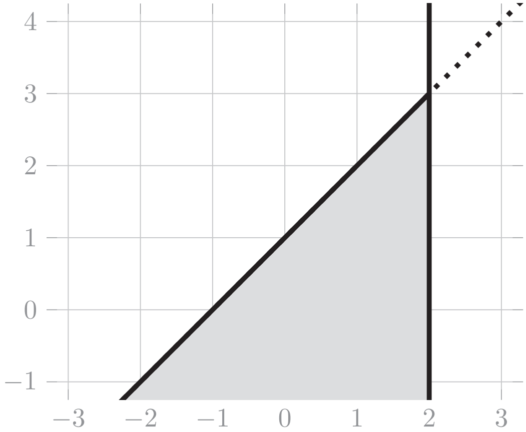
$\mathrm{tconv}(C^\star)$

$$C^\star = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 1 & 0 \end{pmatrix}$$



$\mathrm{tconv}(C)$

$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 4 & 1 & 0 \end{pmatrix}$$



$\mathrm{tconv}(C)$

$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ \infty & 1 & 0 \end{pmatrix}$$

Restricting to polytropes

Problem

A polytrope of a specific combinatorial type might be dual to subdivisions of several different polytopes $\mathcal{N}(\mathcal{Q}_V)$.

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Problem
Not every regular subdivision of $\mathcal{N}(\mathcal{Q}_V)$ *corresponds to a polytrope. Only those weights* $C \in \mathbb{T}^{n \times n}$ *that are Kleene stars.*

Covering relations

Definition

For a given digraph G, the *transitive reduction* G^t is the digraph containing an edge $j \to i$ whenever $j \to i$ is an edge in G and there is no other path between j and i.

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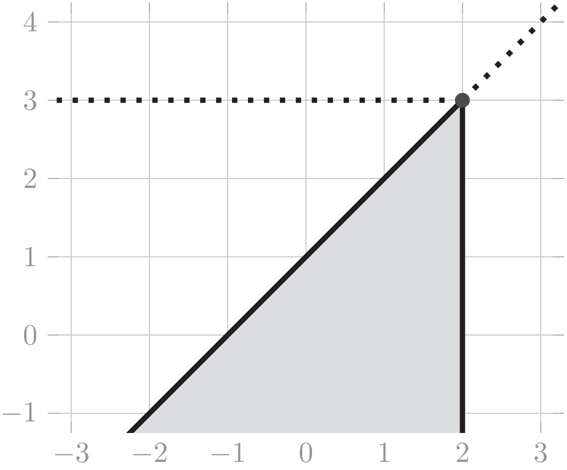
The set of edges E^t of G^t are called the *covering relations* of G.

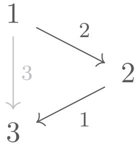
Weighted covering relations

Definition

For a given digraph G, the *weighted transitive reduction* $G^\flat$ is the digraph containing an edge $j \to i$ with weight c_{ij} whenever $j \to i$ is an edge with weight c_{ij} in G and it is the unique shortest path between j and i in G.

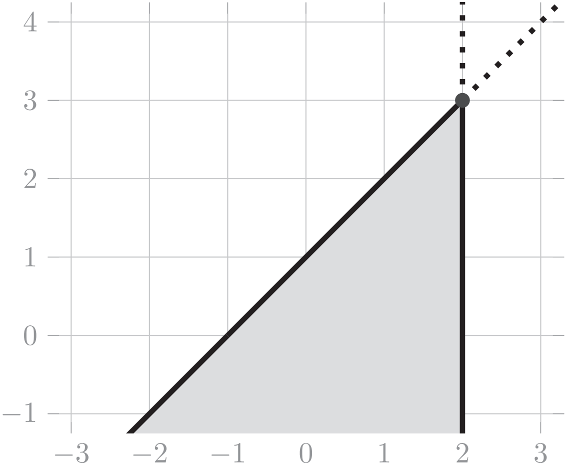
Minimal facet description via covering relations

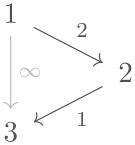




$$A^\star = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 1 & 0 \end{pmatrix}$$

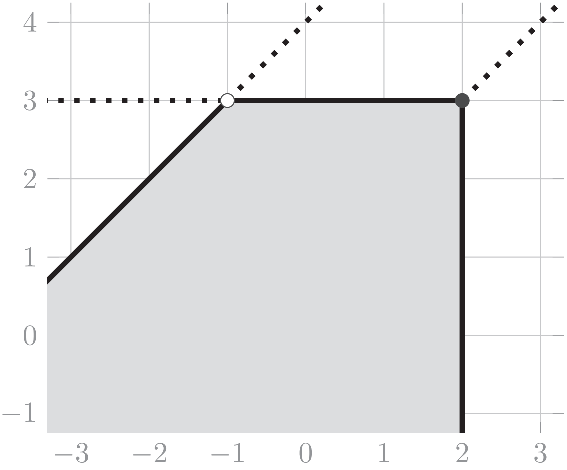
Minimal facet description via covering relations

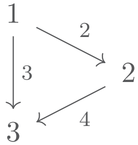




$$A^\flat = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ \infty & 4 & 0 \end{pmatrix}$$

Minimal facet description via covering relations





$$C^\flat = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix}$$

$$C^\flat = C = C^\star$$

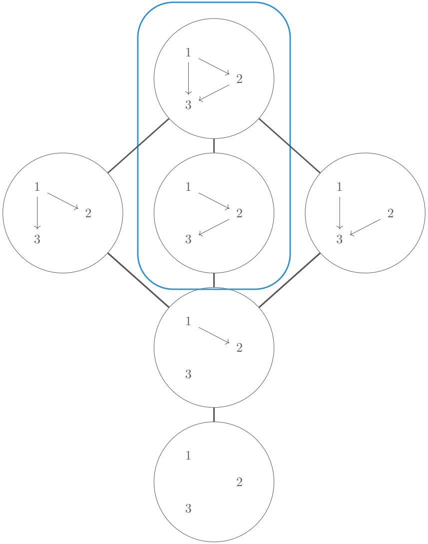
Weighted transitive reduction

Theorem (Améndola and 🚢 2024)

For $C \in \mathbb{T}^{n \times n}$ lower-triangular, the set of matrices $C' \in \mathbb{T}^{n \times n}$ such that $\mathrm{Q}(C) = \mathrm{Q}(C')$ is the polyhedral cone

$$C^{\star} + \mathrm{pos}(e_{ij} \mid j \to i \in G^{\star} \setminus G^{\flat}).$$

Moreover, $C^{\flat}$ is the unique matrix with minimal support such that $\mathrm{Q}(C)$ and $\mathrm{Q}(C^{\flat})$ coincide.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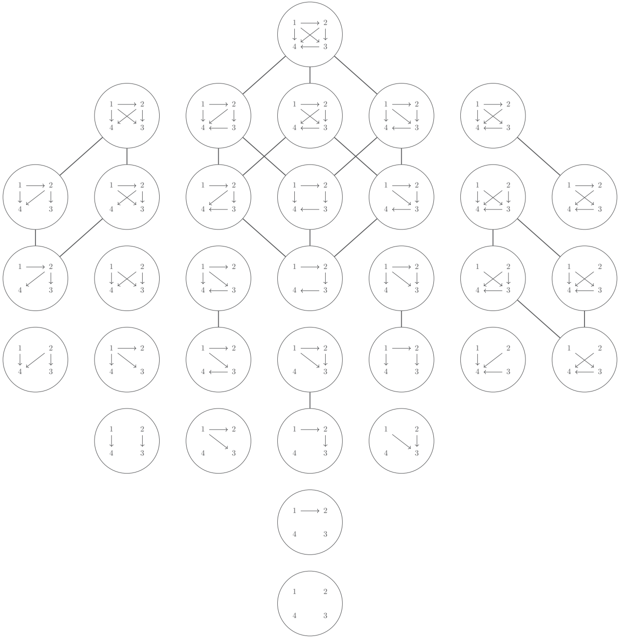


Central subdivisions

Definition

The *fundamental polytope* associated to the digraph G is defined as

$$P_G = \operatorname{conv}\left(\{\, e_i - e_j \mid j \to i \in E(G) \,\} \cup \{\, 0 \,\}\right).$$

Definition

A regular subdivision of P_G is called *central* if 0 is a vertex of every maximal cell.

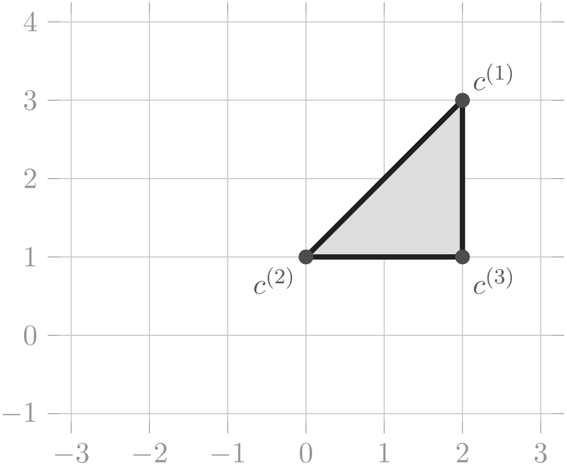
Central subdivisions and polytropes

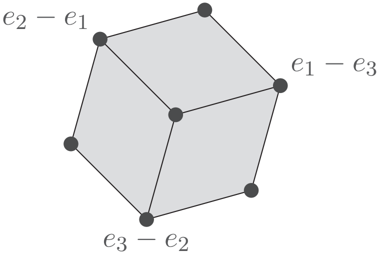
Theorem (Améndola and  2024)

The tropical combinatorial types of full-dimensional polytropes in $\mathbb{TA}^{n-1}$ *are in bijection to the regular central subdivisions of* P_G *where* G *ranges over transitive directed graphs* G *on* n *nodes.*

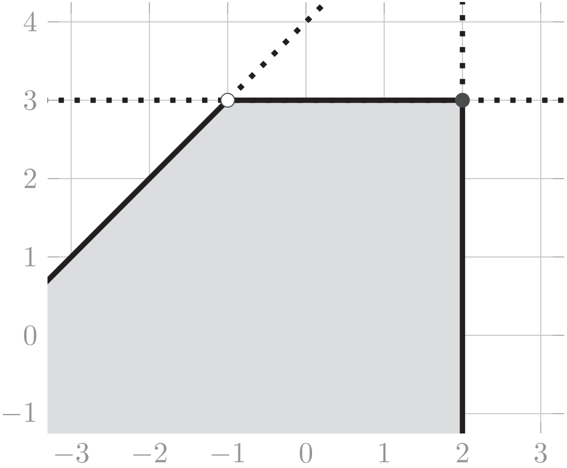
Corollary (Améndola and  2024)

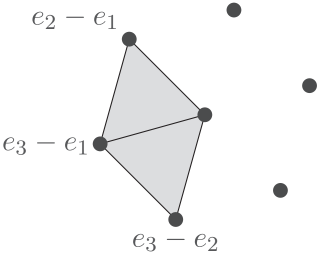
A lower-triangular matrix $C \in \mathbb{T}^{n\times n}$ *supported on* G *induces a central triangulation on* P_G *if and only if* $C = C^\flat$*.*



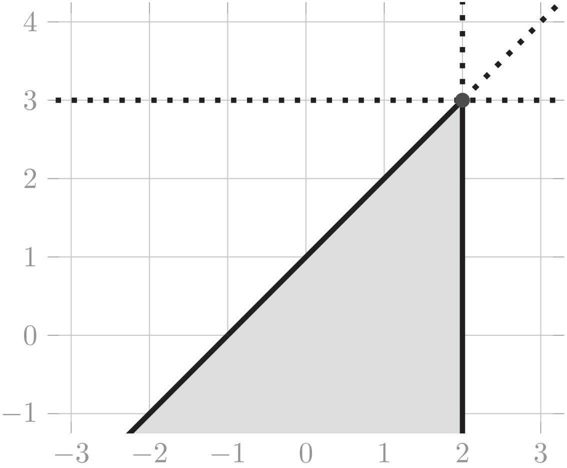


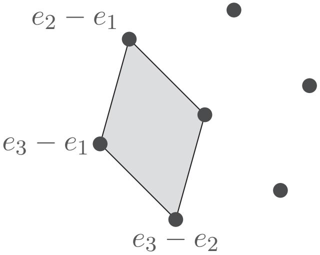
$$C = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$





$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix}$$





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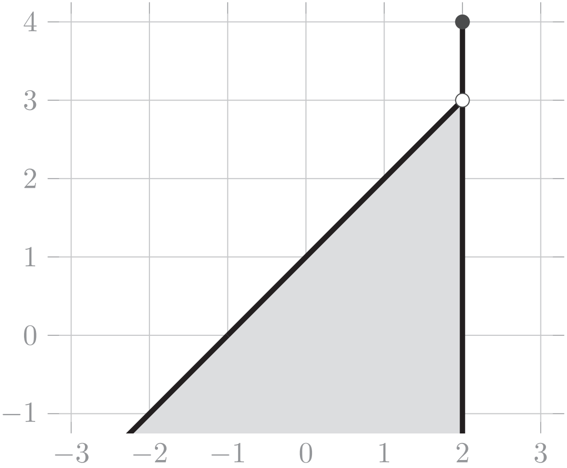
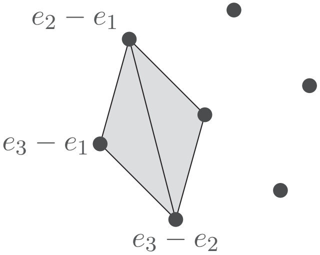


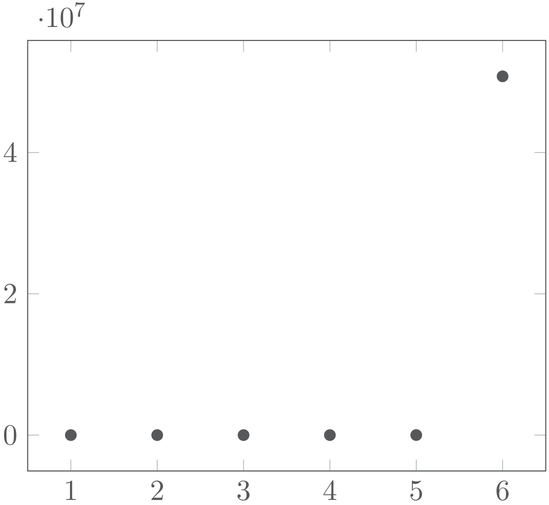


$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 4 & 1 & 0 \end{pmatrix}$$

Enumerating polytropes

n	1	2	3	4	5	6
#combinatorial types	1	2	6	47	6252	50783761
#DAGs	1	2	5	31	302	5984
#transitive DAGs	1	2	5	16	63	318

Enumerating polytropes



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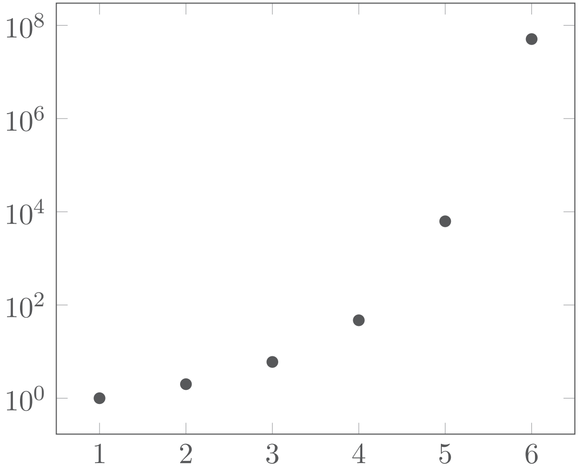


Conditional independence

Markov properties

Definition

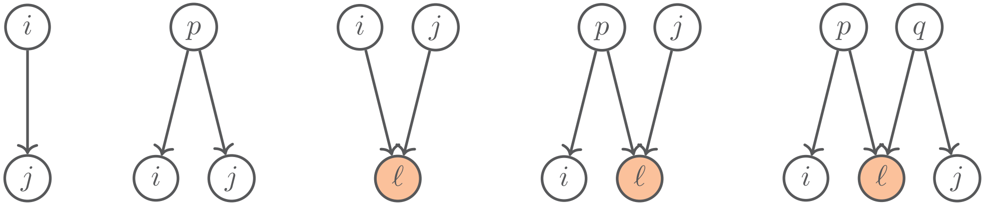
For three disjoint subsets $I, J, K \subset \{1, \ldots, n\}$, we say that I and J are $*$-*separated* by K in a DAG G ($I \perp_* J \mid K$) if there are no $*$-*connecting paths* from I to J in G given K.

Theorem (Améndola-Klüppelberg-Lauritzen-Tran 2022)

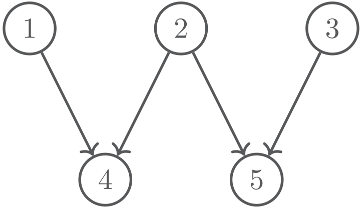
Let X be a max-linear Bayesian network supported on a DAG G. Then, for all $I, J, K \subset \{1, \ldots, n\}$,

$$I \perp_* J \mid K \implies X_I \perp\!\!\!\perp X_J \mid X_K.$$

Types of $*$-connecting paths



If any such path from i to j exists in G, then $i \not\perp_* j \mid K$.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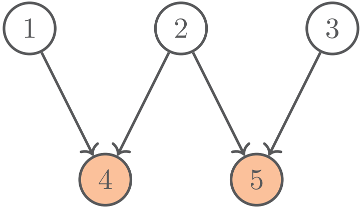
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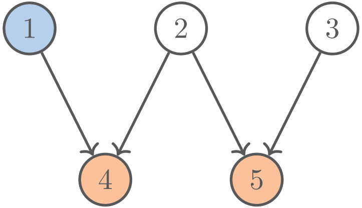
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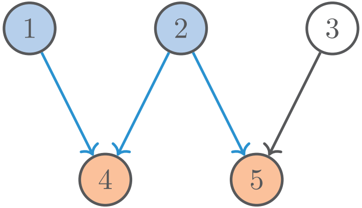
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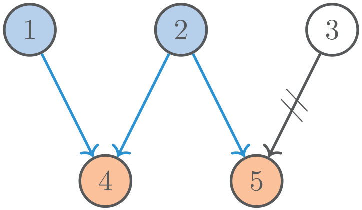
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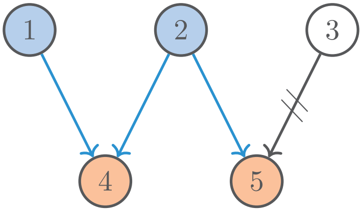
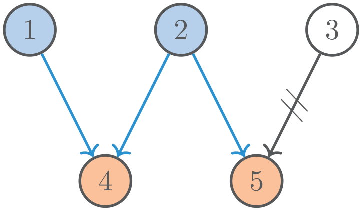


$$\implies 1 \perp_* 3 \mid 4, 5$$



$$\implies 1 \perp_* 3 \mid 4, 5 \implies X_1 \perp\!\!\!\perp X_3 \mid X_4, X_5$$

Maxoids

Definition
Two nodes i and j are *$C^\star$-connected given L* if there exists a shortest path from i to j in G not intersecting L.

Definition
We denote by $\mathcal{M}_*(G,C)$ the set of all triples (I,J,K) for which $I \perp_{C^*} J \mid L$ in the DAG G with weight matrix C and call this the *maxoid* associated to (G,C).

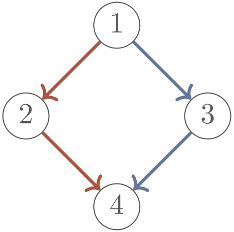
Maxoids

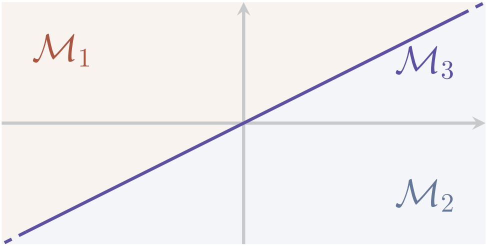
Theorem (Boege, 🚢, Hollering, and Nowell 2025)

For any DAG G *there is a hyperplane arrangement* $\mathcal{H}_G \subseteq \mathbb{R}^E$ *such that for every* $C \in \mathbb{R}^E \setminus \mathcal{H}_G$ *the set*

$$\mathrm{cone}_G(C) := \{\, C' \in \mathbb{R}^E \setminus \mathcal{H}_G \mid \mathcal{M}_*(G, C) = \mathcal{M}_*(G, C') \,\}$$

is a full-dimensional open polyhedral cone.





$\mathcal{M}_1 = \{1 \perp\!\!\!\perp 4 \mid 2,3\ ,\ 1 \perp\!\!\!\perp 4 \mid 2\}$ for $c_{12} + c_{24} < c_{13} + c_{34}$

$\mathcal{M}_2 = \{1 \perp\!\!\!\perp 4 \mid 2,3\ ,\ 1 \perp\!\!\!\perp 4 \mid 3\}$ for $c_{12} + c_{24} > c_{13} + c_{34}$

$\mathcal{M}_3 = \{1 \perp\!\!\!\perp 4 \mid 2,3\ ,\ 1 \perp\!\!\!\perp 4 \mid 2\ ,\ 1 \perp\!\!\!\perp 4 \mid 3\}$ for $c_{12} + c_{24} = c_{13} + c_{34}$.

Compositional graphoids

Semigraphoid: $I \perp\!\!\!\perp J \mid L \wedge I \perp\!\!\!\perp K \mid JL \iff I \perp\!\!\!\perp JK \mid L,$

Intersection: $I \perp\!\!\!\perp J \mid KL \wedge I \perp\!\!\!\perp K \mid JL \implies I \perp\!\!\!\perp JK \mid L,$ and

Composition: $I \perp\!\!\!\perp J \mid L \wedge I \perp\!\!\!\perp K \mid L \implies I \perp\!\!\!\perp JK \mid L.$

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Theorem (Boege, ⛴️, Hollering, and Nowell 2025)

Maxoids are compositional graphoids.

What did we see today?

1. Tropical polytopes, hyperplane arrangements and polytropes
2. How to determine a minimal facet description of a polytrope
3. There are *very many* combinatorial types of polytropes
4. The interplay between conditional independence and the geometry of MLBNs

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