On the Gröbner fans of max-linear Bayesian networks

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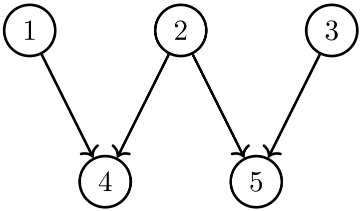
June 2nd, 2025
The 11th Conference of the Fachgruppe Computeralgebra
Leipzig

What is a max-linear Bayesian network?



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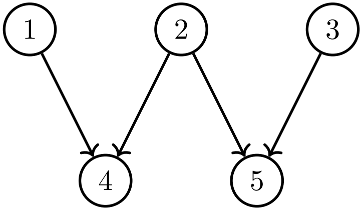
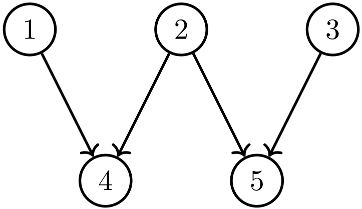


$$X_1 = Z_1$$
$$X_2 = Z_2$$
$$X_3 = Z_3$$
$$X_4 = c_{41}X_1 \vee c_{42}X_2 \vee Z_4$$
$$X_5 = c_{52}X_2 \vee c_{53}X_3 \vee Z_5$$

What is a max-linear Bayesian network?



$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_{41} & c_{42} & 0 & 1 & 0 \\ 0 & c_{52} & c_{53} & 0 & 1 \end{pmatrix} \cdot X \vee Z$$

Max-linear Bayesian networks

Definition (Gissibl and Klüppelberg 2018)

A max-linear Bayesian network is a random vector $X = (X_1, \ldots, X_n)$ where the random variables X_i satisfy the recursive equations

$$X_i = \bigvee_{j=1}^{n} c_{ij} X_j \vee Z_i$$

for some $C \in \mathbb{T}^{n \times n}$ and independent random variables Z_i.

The expression for the X_i can be interpreted as the recursive affine equation

$$X = C \cdot X \vee Z.$$

Markov properties

Definition

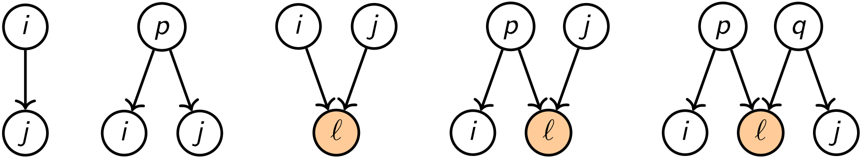
For three disjoint subsets $I, J, K \subset \{1, \ldots, n\}$, we say that I and J are $*$-*separated* by K in a DAG G ($I \perp_* J \mid K$) if there are no $*$-*connecting paths* from I to J in G given K.

Theorem (Améndola, Klüppelberg, Lauritzen and Tran 2022)

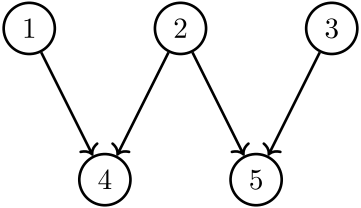
Let X be a max-linear Bayesian network supported on a DAG G. Then, for all $I, J, K \subset \{1, \ldots, n\}$,

$$I \perp_* J \mid K \implies X_I \perp\!\!\!\perp X_J \mid X_K .$$

Types of $*$-connecting paths



If any such path from i to j exists in G, then $i \not\perp_* j \mid K$.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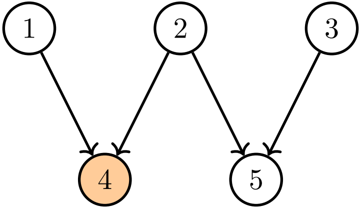
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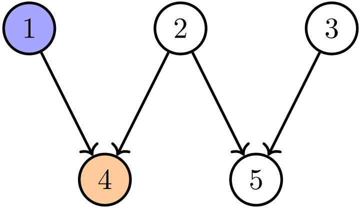
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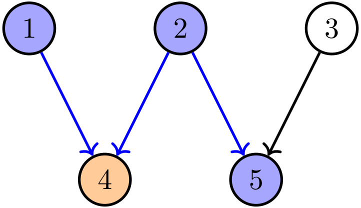
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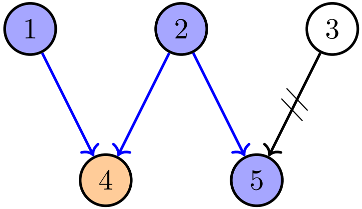
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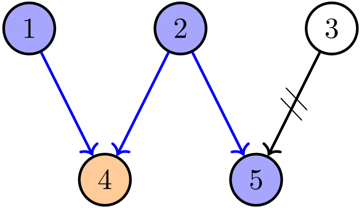
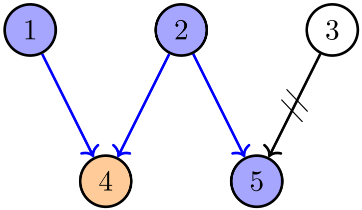


$\implies 1 \perp_* 3 \mid 4$



$$\implies 1 \perp_{*} 3 \mid 4 \implies X_1 \perp\!\!\!\perp X_3 \mid X_4$$

Maxoids

Definition

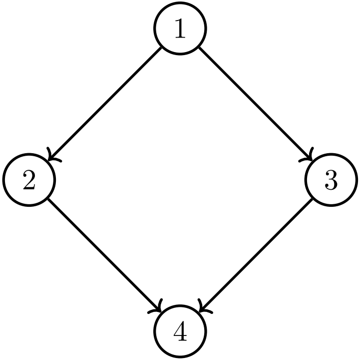
We denote by $\mathcal{M}_*(G, C)$ the set of all triples (I, J, K) for which $I \perp_{C^*} J \mid L$ in the DAG G with weight matrix C and call this the *maxoid* associated to (G, C).

Theorem (Boege, 🛳️, Hollering and Nowell 2025)

For any DAG G there is a hyperplane arrangement $\mathcal{H}_G \subseteq \mathbb{R}^E$ such that for every $C \in \mathbb{R}^E \setminus \mathcal{H}_G$ the set

$$\mathrm{cone}_G(C) := \left\{ C' \in \mathbb{R}^E \setminus \mathcal{H}_G \mid \mathcal{M}_*(G, C) = \mathcal{M}_*(G, C') \right\}$$

is a full-dimensional open polyhedral cone. The collection of all closures of such cones for a fixed G forms a complete polyhedral fan $\mathcal{F}_G$ in $\mathbb{R}^E$. Moreover the map which sends a cone of $\mathcal{F}_G$ to its maxoid is an inclusion-reversing surjection.



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$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_{21} & 1 & 0 & 0 \\ c_{31} & 0 & 1 & 0 \\ 0 & c_{42} & c_{43} & 1 \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \vee \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}$$

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$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 0 & \infty & \infty & \infty \\ c_{21} & 0 & \infty & \infty \\ c_{31} & \infty & 0 & \infty \\ c_{42}c_{21} \oplus c_{43}c_{31} & c_{42} & c_{43} & 0 \end{pmatrix} \odot \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}$$

Kleene stars and polytropes

Definition

The Kleene star $C^\star$ of a tropical matrix $C \in \mathbb{T}^{n\times n}$ is given by

$$C^\star := I_n \oplus C \oplus \cdots \oplus C^{\odot(n-1)} \oplus \ldots.$$

Theorem (de la Puente 2013)

For a tropical matrix $C \in \mathbb{T}^{n\times n}$, the following sets coincide:

- *The (tropical) column span of the Kleene star $C^\star$.*
- *The intersection $\mathrm{Q}(C)$ of half-spaces $x_i - x_j \leq c_{ij}$.*

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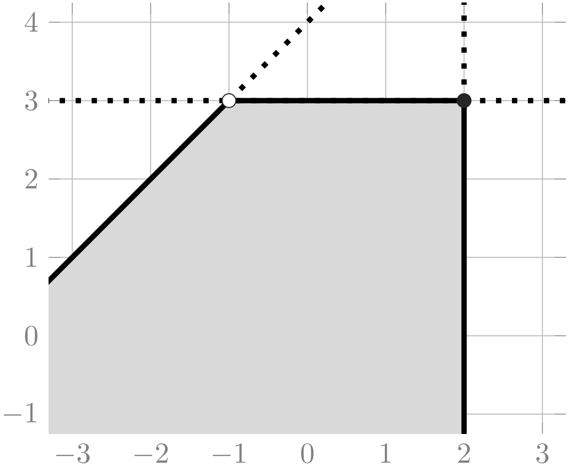
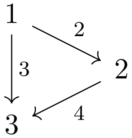
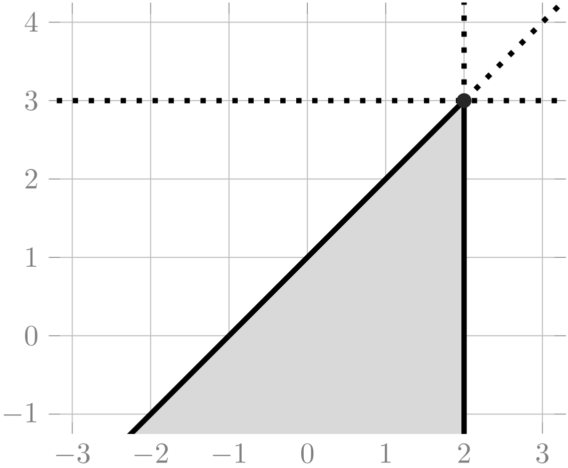
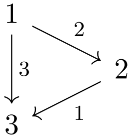
The polyhedron $\mathrm{Q}(C)$ is the *polytrope* associated to C.



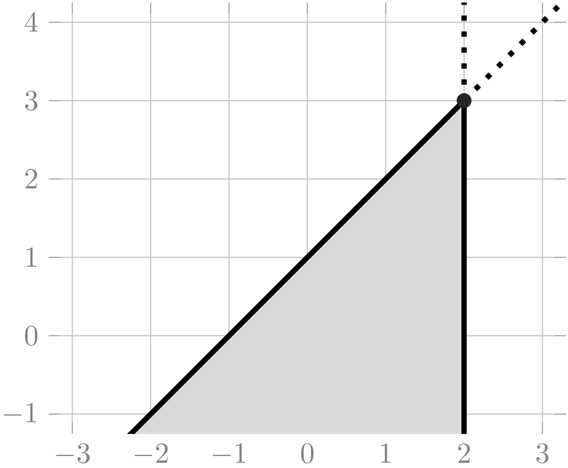


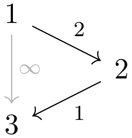
$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix}$$





$$C^* = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 1 & 0 \end{pmatrix}$$





$$C^\flat = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ \infty & 1 & 0 \end{pmatrix}$$

Edge rings and edge ideals

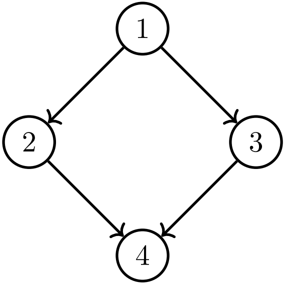
Definition

The *edge ring* of a directed graph G, denoted by $\mathbb{k}[G]$, is the polynomial ring generated by the edges of G, that is

$$\mathbb{k}[G] := \mathbb{k}[\{e_{ij} \mid i \to j \text{ is a directed edge in } G\}].$$

The *edge ideal* $I_G \trianglelefteq \mathbb{k}[G]$ is the toric ideal given by the kernel of the ring morphism

$$\pi\colon \mathbb{k}[G] \to \mathbb{k}[s_1^{\pm}, t_1^{\pm}, \dots, s_n^{\pm}, t_n^{\pm}],\ \ e_{ij} \mapsto \frac{s_i}{t_j}.$$



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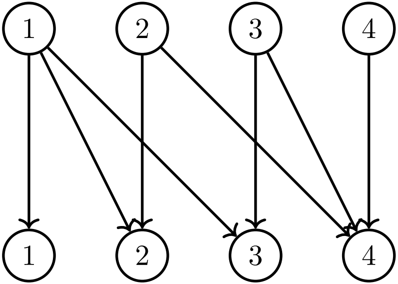

$B(G)$

… and edge ideals

Definition

To an even closed walk $w = i_0 \to i_1 \to \cdots \to i_\ell$ in a graph $B(G)$ we associate a binomial

$$e^w := e_1 \dots e_{r-1} - e_2 \dots e_\ell \in \mathbb{k}[G]$$

where $e_k = e_{i_{k-1} i_k}$.

Theorem (Reyes, Tatakis and Thoma 2012)

A universal Gröbner basis of the edge ideal I_G is given by the set

$$\{\, e^w \mid e^w \text{ is primitive and } w \text{ is an even closed walk.} \,\}.$$

The elimination point-of-view

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Proposition

Let $H \subseteq G$ be directed graphs. Then

$$I_H = I_G \cap \Bbbk[H].$$

In other words, I_H is the elimination ideal of I_G for $i \to j \in G \setminus H$.

The determinantal point-of-view

Proposition (Villarreal 2015)

Let $G = K_n$ be the complete digraph. Then, I_G is the (dehomogenized) ideal of the Segre embedding of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. In other words, I_G is the determinantal ideal given by the 2×2-minors of

$$\begin{pmatrix} 1 & e_{12} & \dots & e_{1n} \\ e_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e_{(n-1)n} \\ e_{n1} & \dots & e_{n(n-1)} & 1 \end{pmatrix}.$$

Polytropes and edge ideals

$$\mathrm{Q}(C) = \{\, x \in \mathbb{R}^n \mid x_i - x_j \leq c_{ij} \,\} \quad \text{for } C \in \mathbb{T}^{n \times n}$$

Theorem (⛴️ 2025)

A $(x_i - x_j = c_{ij})$-hyperplane defines a facet of $\mathrm{Q}(C)$ if and only if e_{ij} is not contained in the linear part of $\mathrm{in}_C(I_G)$. In this case, $C = C^\star$.

Polytropes and edge ideals

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For C generic, this is an equivalence.

Polytropes and edge ideals

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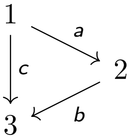
For C generic, this is an equivalence.

Corollary (🚢 2025)

Suppose e_{ij} is in the linear part of $\mathrm{in}_C(I_G)$. Then, $\mathrm{Q}(C) = \mathrm{Q}(C')$ where C' agrees with except for $c'_{ij} = \infty$.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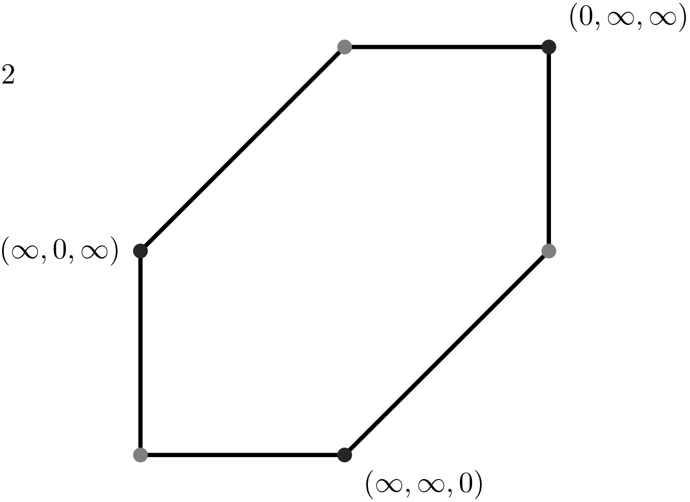
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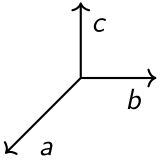
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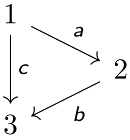
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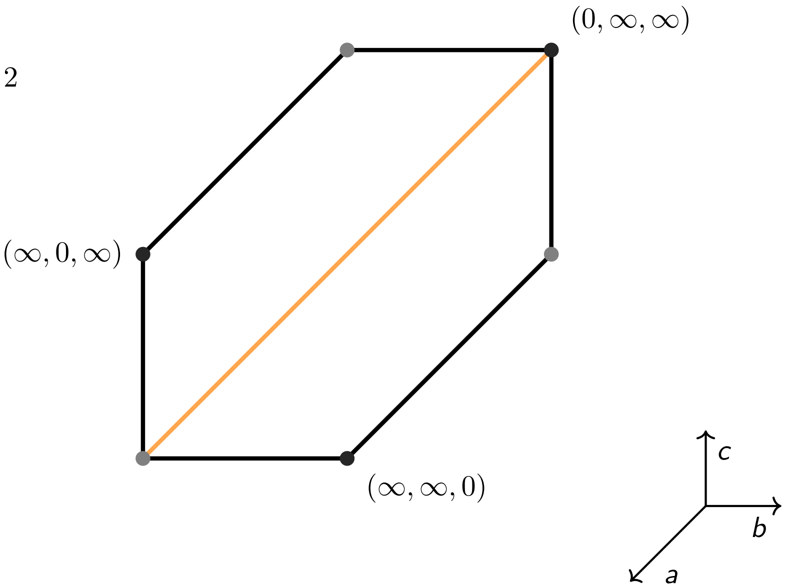
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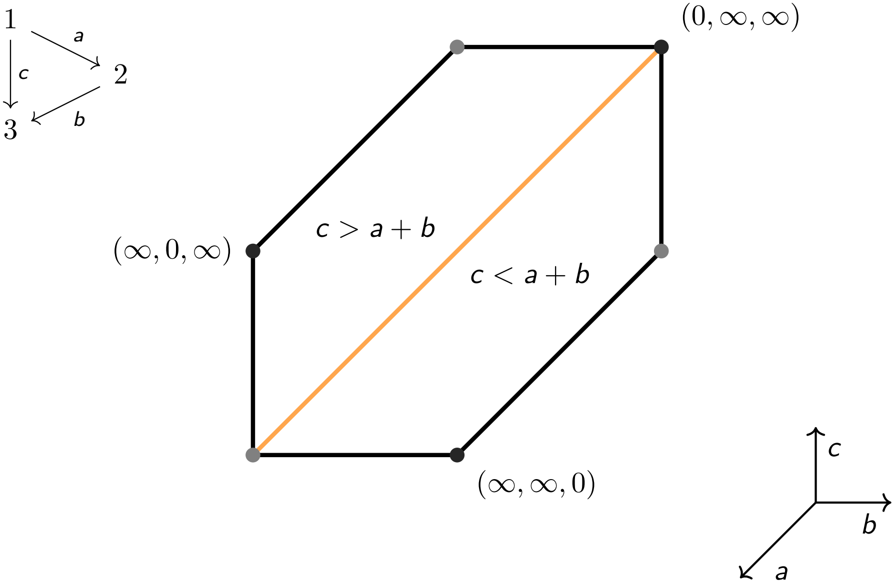
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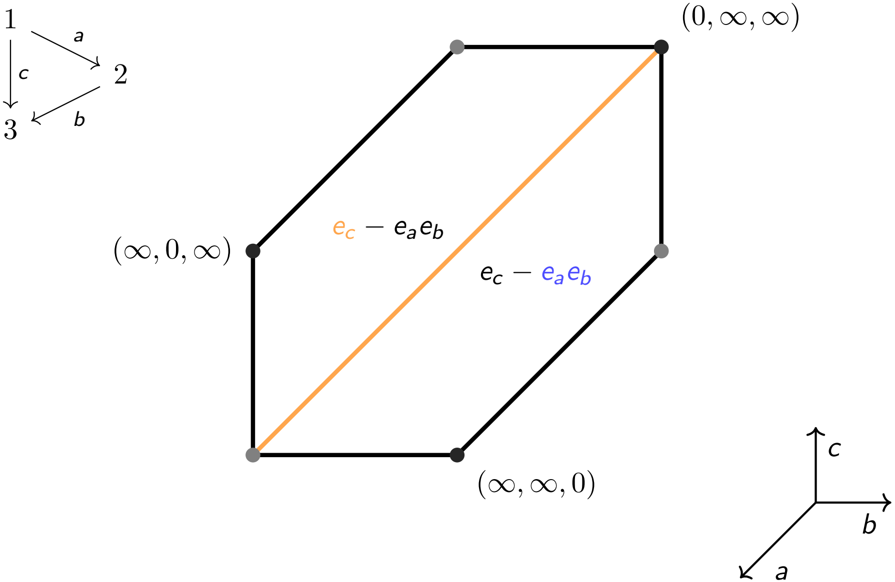
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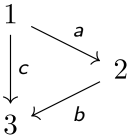

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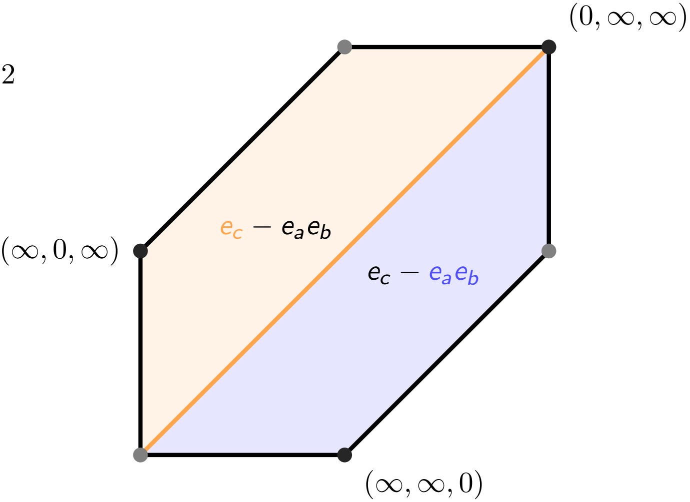

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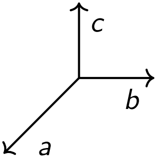

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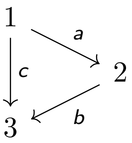
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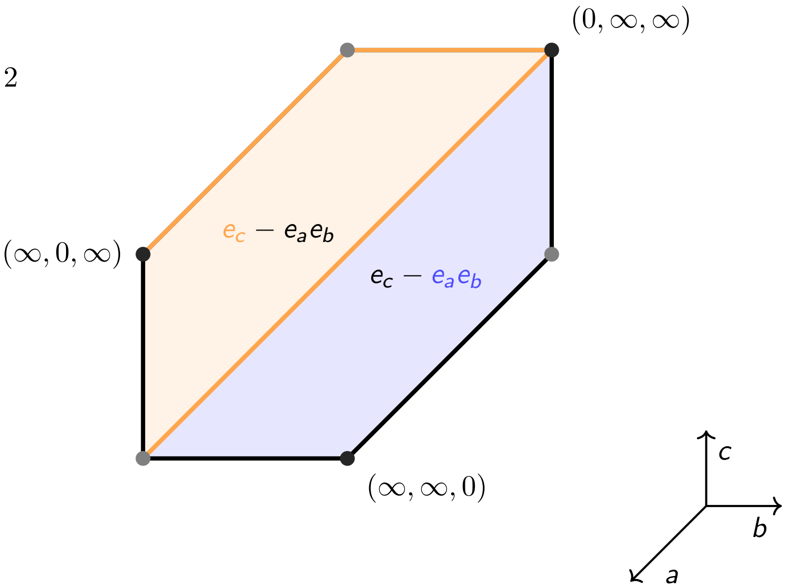
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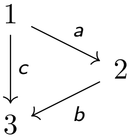
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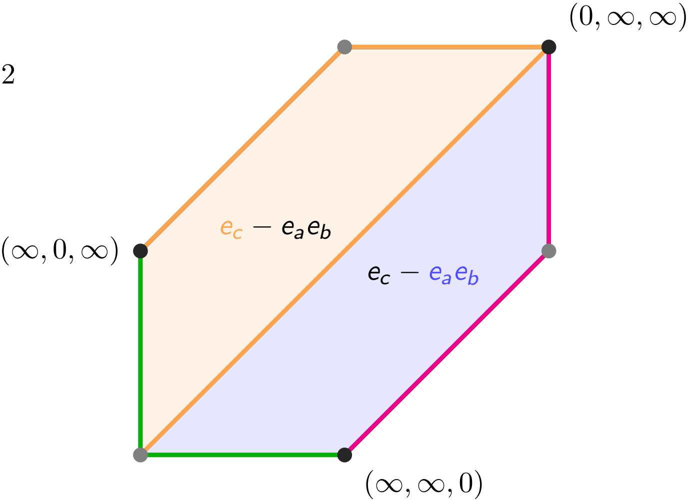
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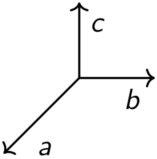
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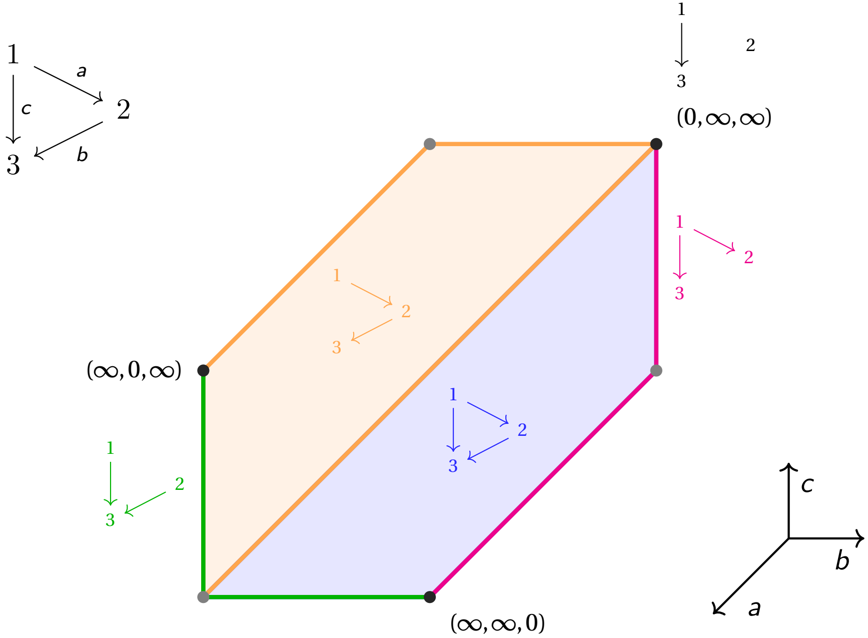

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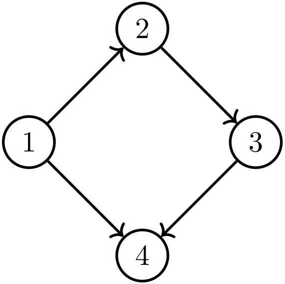
Caveat emptor



The edge ideal of this graph is generated by the binomial

$$e_{14} - e_{12}e_{23}e_{34}$$

which is *not* the 2×2-minor of a matrix.

Proof: The elimination point-of-view

Proposition

Let $H \subseteq G$ be directed graphs. Then

$$I_H = I_G \cap \mathbb{k}[H].$$

Proof sketch: I_G can be taken as the ideal generated by e^w for all even closed walks in G. I_H is then generated by those e^w for which w is an even closed walk in $H \subseteq G$.

Proof: The elimination point-of-view

Proposition

Let $H \subseteq G$ be directed graphs. Then

$$I_H = I_G \cap \mathbb{k}[H].$$

In other words, I_H is the elimination ideal of I_G for $i \to j \in G \setminus H$.

Proof sketch: I_G can be taken as the ideal generated by e^w for all even closed walks in G. I_H is then generated by those e^w for which w is an even closed walk in $H \subseteq G$.