Likelihood Geometry of Max-Linear Bayesian Networks

and why this is a story about tropical polytopes

Kamillo Ferry

Technische Universität Berlin

July 17th, 2024

Tropical numbers

Definition ([4])

The *min-plus tropical semiring* is $\mathbb{T} := \mathbb{T}_{\text{min}} := (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ where $\oplus :=$ min and $\odot := +$.

Alternatively, $\mathbb{T}_{\text{max}} := (\mathbb{R} \cup \{-\infty\}, \text{max}, +)$, $(\mathbb{R}_{\geq 0}, \text{min}, \cdot)$, and $(\mathbb{R}_{\geq 0}, \text{max}, \cdot)$ are equally valid choices for tropical semirings.

$$
\begin{array}{ccc}
\mathbb{T}_{\text{min}} & \xleftrightarrow{\;\cdot(-1)\;} & \mathbb{T}_{\text{max}} \\
\text{log}\uparrow \;\downarrow \textit{exp} & & \text{log}\uparrow \;\downarrow \textit{exp} \\
(\mathbb{R}_{\geq 0}, \text{min}) & \xleftrightarrow{\;\frac{1}{\cdot}\;} & (\mathbb{R}_{\geq 0}, \text{max})
\end{array}
$$

Tropical affine and projective space

Definition

Tropical affine space refers to $\mathbb{TA}^{d-1} := \mathbb{R}^d / \mathbb{R}\mathbb{1}$ while *tropical projective space* is defined as $\mathbb{TP}^{d-1} := \mathbb{T}^d \setminus \{(\infty, \dots, \infty)\} / \mathbb{R}\mathbb{1}$.

We can identify $\mathbb{TA}^{d-1}$ with Euclidean space $\mathbb{R}^{d-1}$ via

$$(x_1, \dots, x_n) \cong (0, x_2 - x_1, \dots, x_n - x_1).$$

Tropical polytope and polytropes

Definition

The *tropical convex hull* of $V \subset \mathbb{TP}^{d-1}$ finite is the *(min-plus)-linear span* of V, i. e.

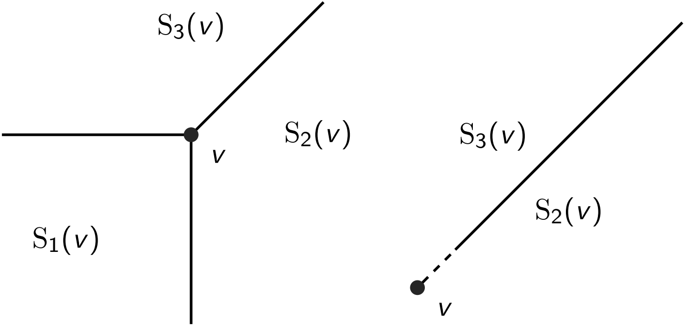
$$\operatorname{tconv}(V) := \left\{ \lambda_1 \odot v^{(1)} \oplus \cdots \oplus \lambda_n \odot v^{(n)} \;\middle|\; \lambda_i \in \mathbb{T}, v^{(j)} \in V \right\}.$$

Note: A tropical polytope is the column span of a matrix with the vertices in V as columns.

Definition ([3])

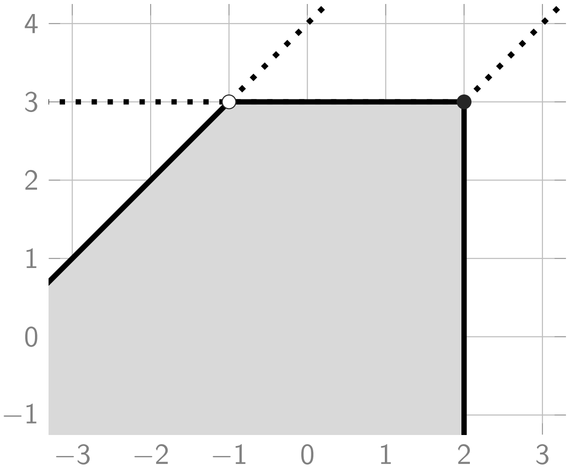
A tropical polytope P is called a polytrope if it is classically convex.

Tropical hyperplanes



$$\alpha_v = \max\{\, x_1 - v_1, x_2 - v_2, x_3 - v_3 \,\}$$

Tropical hyperplane arrangements and polytopes



$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix}$$

Max-linear Bayesian networks

Definition ([2])

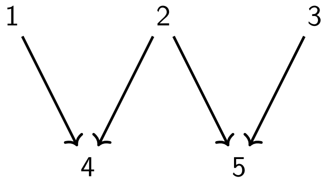
A max-linear Bayesian network is a random vector $X = (X_1, \ldots, X_N)$ where the random variables X_i can be recursively expressed as

$$X_i = \bigvee_{j=1}^{n} c_{ij} X_j \vee Z_i.$$

The expression for the X_i can be interpreted as the recursive affine equation

$$X = C \cdot X \vee Z.$$

Putting the 'graph' in graphical model



$$X_1 = Z_1$$
$$X_2 = Z_2$$
$$X_3 = Z_3$$
$$X_4 = c_{41}X_1 \vee c_{42}X_2 \vee Z_4$$
$$X_5 = c_{52}X_2 \vee c_{53}X_3 \vee Z_5$$

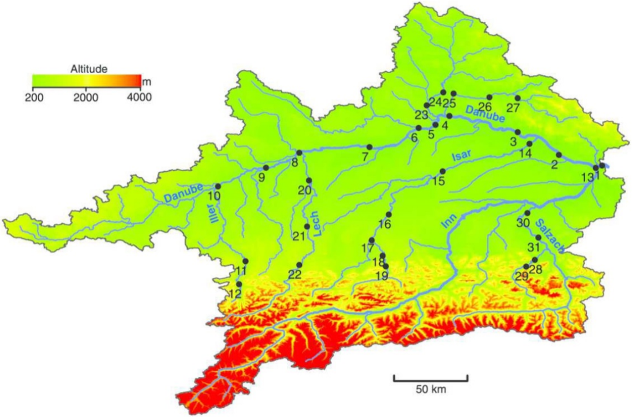
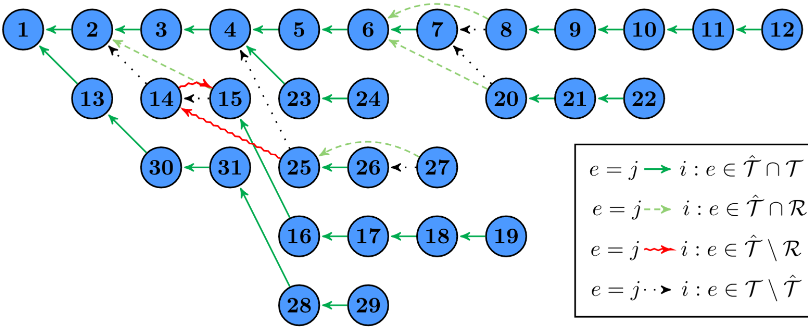


Fig. 3. Topographic map of the Upper Danube Basin, showing sites of the 31 gauging stations along the Danube and its tributaries.



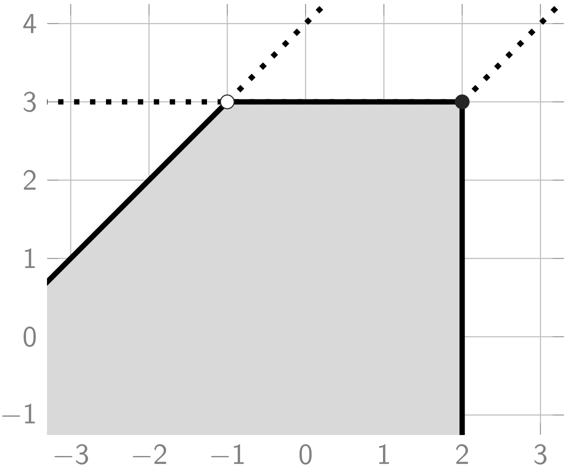
$e = j \longrightarrow i : e \in \hat{\mathcal{T}} \cap \mathcal{T}$

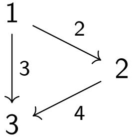
$e = j \dashrightarrow i : e \in \hat{\mathcal{T}} \cap \mathcal{R} \setminus \mathcal{T}$

$e = j \rightsquigarrow i : e \in \hat{\mathcal{T}} \setminus \mathcal{R}$

$e = j \cdots\!\!> i : e \in \mathcal{T} \setminus \hat{\mathcal{T}}$

Tropical hyperplane arrangements and polytopes





$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix}$$

Networks and shortest-paths

The expression for the X_i can be interpreted as the affine expression

$$X = C \cdot X \vee Z.$$

Since $C^k = C^{k+1}$ for $k \gg 0$, we can solve this recursive equation to get the expression

$$X = \underbrace{\left(I_n \vee C \vee \cdots \vee C^{n-1}\right)}_{=:C^\star} \cdot Z.$$

The matrix $C^\star$ is called the *Kleene star*.

Arrangements and polytropes from digraphs

Definition

For a matrix $C \in \mathbb{T}^{n \times n}$ its *weighted digraph polyhedron* $\mathrm{Q}(C)$ is given by the (ordinarily convex) set

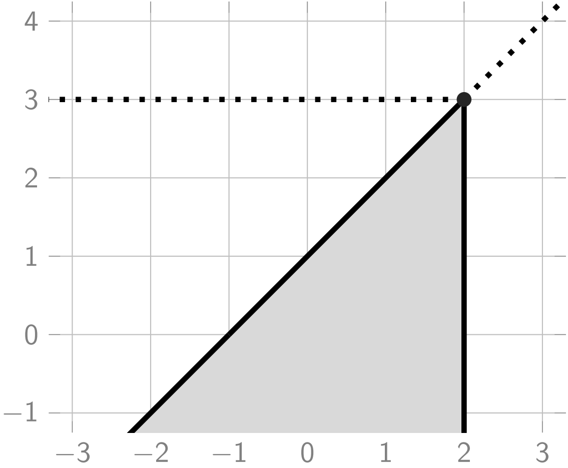
$$\mathrm{Q}(C) = \{ x \in \mathbb{TP}^{n-1} \mid x_i - x_j \leq c_{ij} \text{ for all } 1 \leq i, j \leq n, i \neq j \}.$$

Theorem ([1, 5])

Let P be a polytrope. Then, there exists $C \in \mathbb{T}^{n \times n}$ such that

$$P = \mathrm{tconv}(C^{\star}) = \mathrm{Q}(C) = \mathrm{Q}(C^{\star}).$$

Finding a minimal facet description



$$C^{\star} = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 6 & 4 & 0 \end{pmatrix}$$

Covering relations

Definition

For a given digraph $\mathcal{D}$, the *transitive reduction* $\mathcal{D}^t$ is the digraph containing an edge $j \to i$ whenever $j \to i$ is an edge in $\mathcal{D}$ and there is no other path between j and i.

The set of edges E^t of $\mathcal{D}^t$ are called the *covering relations* of $\mathcal{D}$.

Weighted covering relations

Definition

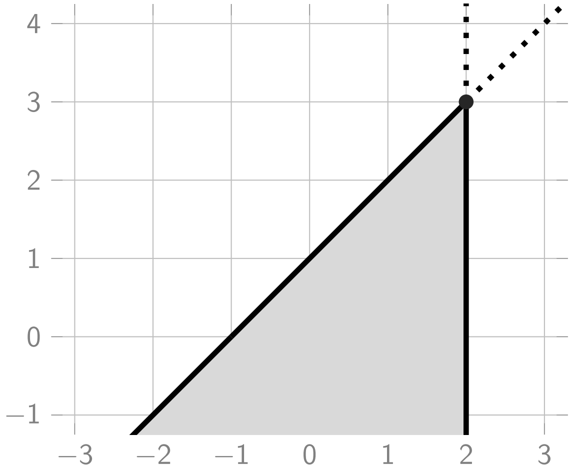
For a given digraph $\mathcal{D}$, the *weighted transitive reduction* $\mathcal{D}^\flat$ is the digraph containing an edge $j \to i$ with weight c_{ij} whenever $j \to i$ is an edge with weight c_{ij} in $\mathcal{D}$ and it is the unique shortest path between j and i in $\mathcal{D}$.

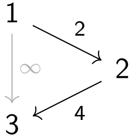
The set of edges $E^\flat$ of $\mathcal{D}^\flat$ are called the *weighted covering relations* of $\mathcal{D}$.

Theorem

Let $C \in \mathbb{T}^{n\times n}$ be lower triangular. Denote by $C^\flat$ the weight matrix of $\mathcal{D}^\flat$. Then $\mathrm{Q}(C^\flat) = \mathrm{Q}(C)$ and $C^\flat$ is minimal with respect to number of finite entries.

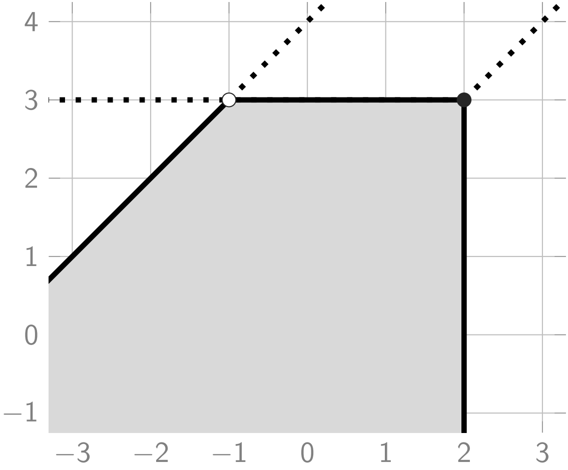
Minimal facet description via covering relations

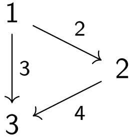




$$C^\flat = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ \infty & 4 & 0 \end{pmatrix}$$

Minimal facet description via covering relations





$$C^\flat = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix}$$

$$C^\flat = C = C^\star$$

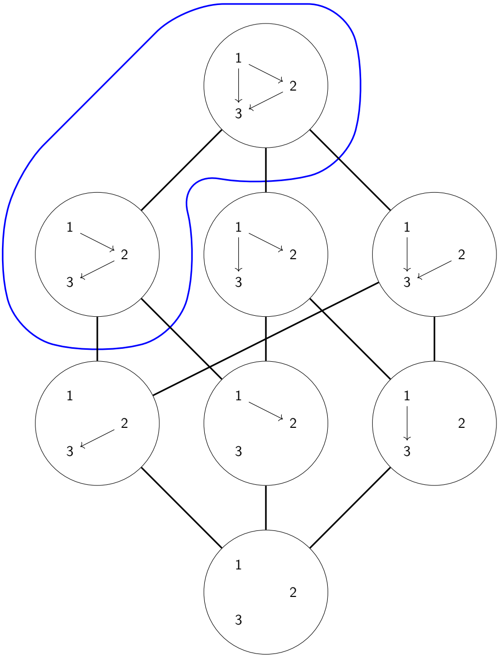
Lower and upper bounds

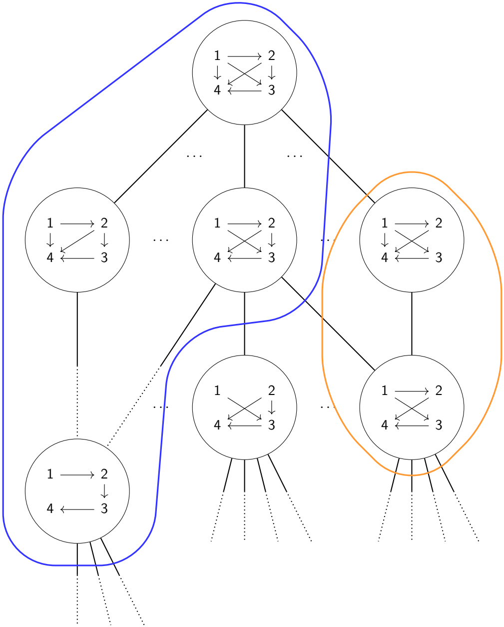
Proposition

The unweighted transitive reduction $\mathcal{D}^t$ gives a lower bound for $\mathcal{D}^\flat$.

Proposition

The Kleene star $C^\star$ gives an upper bound for edges that can be added without affecting $\mathrm{Q}(C)$.


1
2
3
1
2
3
1
2
3
1
2
3
1
2
3
1
2
3
1
2
3
1
2
3


1 2 4 3
1 2 4 3
1 2 4 3
1 2 4 3
1 2 4 3
1 2 4 3
1 2 4 3
...
...
...
...
...
...

Bibliography I

[1] Mike Develin and Bernd Sturmfels. "Tropical convexity". In: *Documenta Mathematica* 9 (2004), pp. 1–27. ISSN: 1431-0635, 1431-0643. DOI: 10.4171/dm/154.

[2] Nadine Gissibl and Claudia Klüppelberg. "Max-linear models on directed acyclic graphs". In: *Bernoulli* 24.4 (1st Nov. 2018), pp. 2693–2720. ISSN: 1350-7265. DOI: 10.3150/17-BEJ941.

[3] Michael Joswig and Katja Kulas. "Tropical and ordinary convexity combined". In: *Advances in Geometry* 10.2 (1st Apr. 2010). Publisher: De Gruyter, Section: Advances in Geometry, pp. 333–352. ISSN: 1615-7168. DOI: 10.1515/advgeom.2010.012.

[4] Diane Maclagan and Bernd Sturmfels. *Introduction to Tropical Geometry*. Vol. 161. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, 15th Apr. 2015. DOI: 10.1090/gsm/161.

Bibliography II

[5] Ngoc Mai Tran. "Enumerating polytropes". In: *Journal of Combinatorial Theory, Series A* 151 (1st Oct. 2017), pp. 1–22. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2017.03.011.

[6] Ngoc Mai Tran, Johannes Buck and Claudia Klüppelberg. "Estimating a directed tree for extremes". In: *Journal of the Royal Statistical Society Series B: Statistical Methodology* 86.3 (12th July 2024), pp. 771–792. ISSN: 1369-7412, 1467-9868. DOI: 10.1093/jrsssb/qkad165.