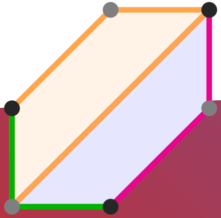
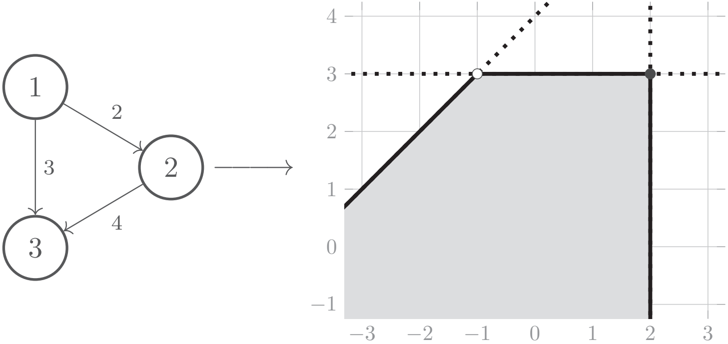


Tropical combinatorics of max-linear Bayesian networks

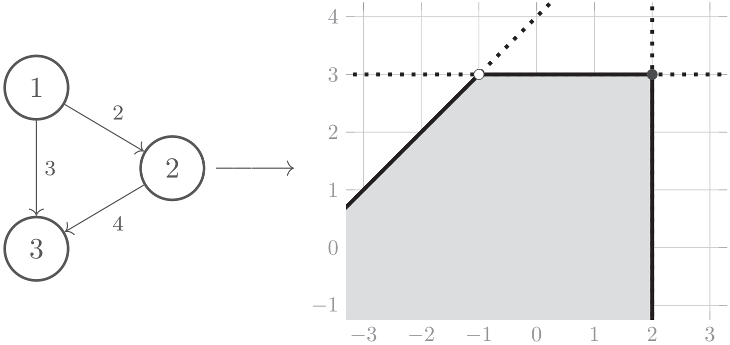
Kamillo Ferry (🚢)

for the SIAM Conference on Applied Algebraic Geometry (AG25)
July 10, 2025

Setup (the ‘TL;DR’)



Setup (the 'TL;DR')



Question

How can we characterize the combinatorics of polyhedra like the one above that come from (labeled) weighted acyclic digraphs?

What is a MLBN?

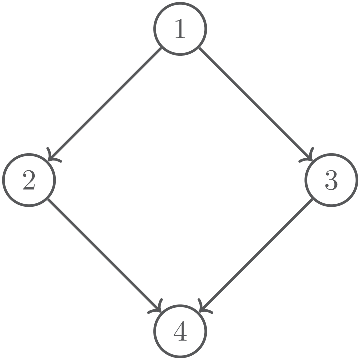
Definition

A max-linear Bayesian network is a random vector $X = (X_1, \ldots, X_n)$ where the random variables X_i satisfy the recursive equations

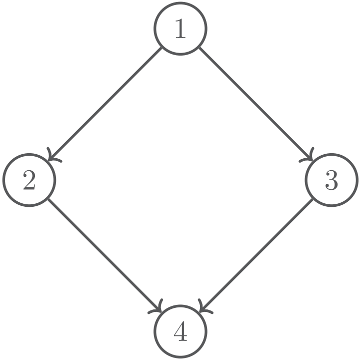
$$X_i = \bigvee_{j=1}^{n} c_{ij} X_j \vee Z_i$$

for some $C \in \mathbb{T}^{n \times n}$ and $Z_1, \ldots, Z_n$ i. i. d. random variables.

The first contact

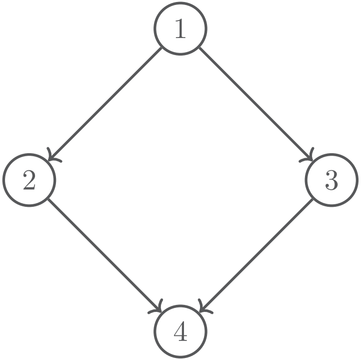


The first contact



$$X_1 = Z_1$$
$$X_2 = c_{12}X_1 \vee Z_2$$
$$X_3 = c_{13}X_1 \vee Z_3$$
$$X_4 = c_{24}X_2 \vee c_{34}X_3 \vee Z_4$$

The first contact

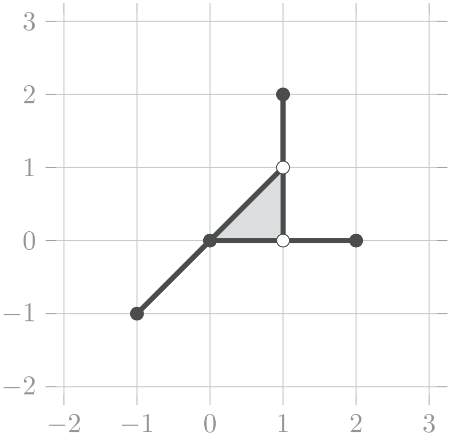


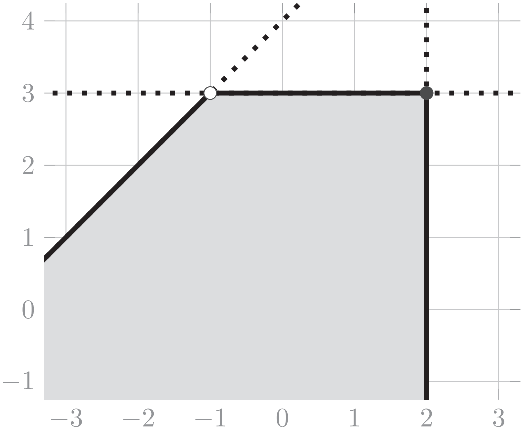
$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_{12} & 0 & 0 & 0 \\ c_{13} & 0 & 0 & 0 \\ 0 & c_{24} & c_{34} & 0 \end{pmatrix} \cdot X \vee Z$$

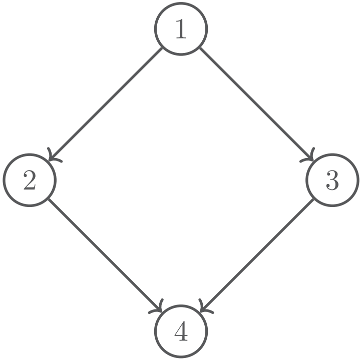
Definition

The *tropical convex hull* of $V \in \mathbb{T}^{d\times n}$ finite is the *(min-plus)-linear span* of V, i.e.

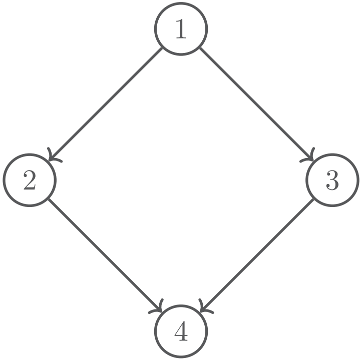
$$\operatorname{tconv}(V) := \left\{ \lambda_1 \odot v^{(1)} \oplus \cdots \oplus \lambda_n \odot v^{(n)} \;\middle|\; \lambda_i \in \mathbb{R} \right\}.$$


3
2
1
0
−1
−2
−2
−1
0
1
2
3


4
3
2
1
0
−1
−3
−2
−1
0
1
2
3



$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_{12} & 0 & 0 & 0 \\ c_{13} & 0 & 0 & 0 \\ 0 & c_{24} & c_{34} & 0 \end{pmatrix} \cdot X \vee Z$$



$$X = \begin{pmatrix} \infty & \infty & \infty & \infty \\ c_{12} & \infty & \infty & \infty \\ c_{13} & \infty & \infty & \infty \\ \infty & c_{24} & c_{34} & \infty \end{pmatrix} \odot X \oplus Z$$

$$
\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} \infty & \infty & \infty & \infty \\ c_{12} & \infty & \infty & \infty \\ c_{13} & \infty & \infty & \infty \\ \infty & c_{24} & c_{34} & \infty \end{pmatrix} \odot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \oplus \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}
$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \infty & \infty & \infty \\ c_{12} & 0 & \infty & \infty \\ c_{13} & \infty & 0 & \infty \\ c_{12}c_{24} \oplus c_{13}c_{34} & c_{24} & c_{34} & 0 \end{pmatrix}}_{=:C^\star} \odot \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}$$

Tropical polytopes and polytropes

Note: The matrix

$$C^\star = I_n \oplus A \oplus A^2 \oplus \cdots \oplus A^n \oplus \dots$$

is called the *Kleene star* of C and records the shortest paths in a graph with weight matrix C.

Definition (Joswig and Kulas 2010)

A tropical polytope P is called a polytrope if it is classically convex.

Tropical polytopes and polytropes

Note: The matrix

$$C^{\star} = I_n \oplus A \oplus A^2 \oplus \cdots \oplus A^n \oplus \dots$$

is called the *Kleene star* of C and records the shortest paths in a graph with weight matrix C.

Theorem (Butkovič 2010; Joswig and Loho 2016; Puente 2013)

A tropical polytope $P \subset \mathbb{TA}^{d-1}$ is a polytrope if and only if

$$P = \operatorname{tconv}(C^{\star}) = \mathrm{Q}(C)$$

for some $C \in \mathbb{T}^{n \times n}$.

Polyhedra from weighted digraphs

Definition

For $C \in \mathbb{T}^{n\times n}$, the *weighted digraph polyhedron* $\mathrm{Q}(C)$ is defined by the linear inequalities

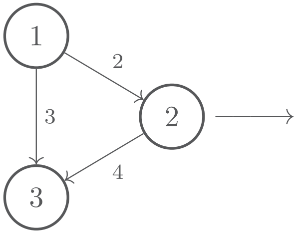
$$x_i - x_j \leq c_{ij} .$$

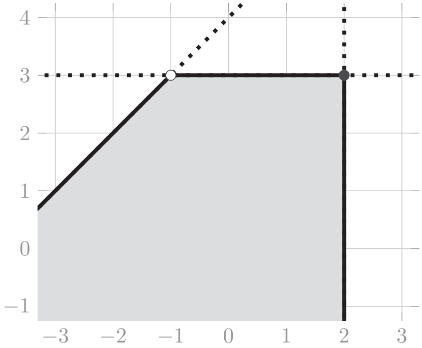
Polyhedra from weighted digraphs

Definition

For $C \in \mathbb{T}^{n \times n}$, the *weighted digraph polyhedron* $\mathrm{Q}(C)$ is defined by the linear inequalities

$$x_i - x_j \leq c_{ij} .$$





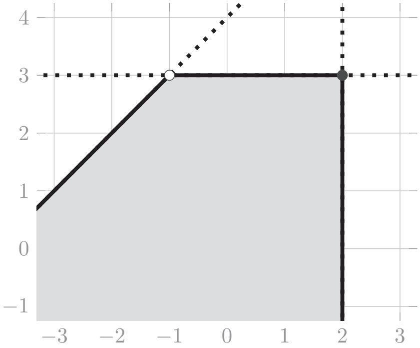
Polyhedra from weighted digraphs

Definition

For $C \in \mathbb{T}^{n \times n}$, the *weighted digraph polyhedron* $\mathrm{Q}(C)$ is defined by the linear inequalities

$$x_i - x_j \leq c_{ij}\,.$$

$$\begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix} \longrightarrow$$



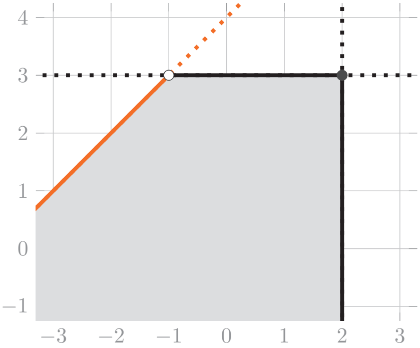
Polyhedra from weighted digraphs

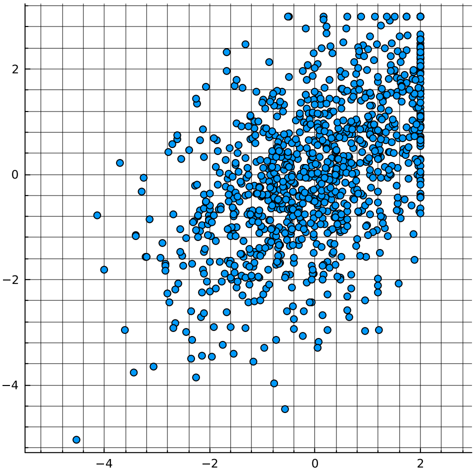
Definition

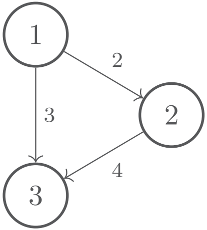
For $C \in \mathbb{T}^{n\times n}$, the *weighted digraph polyhedron* $\mathrm{Q}(C)$ is defined by the linear inequalities

$$x_i - x_j \leq c_{ij}\,.$$

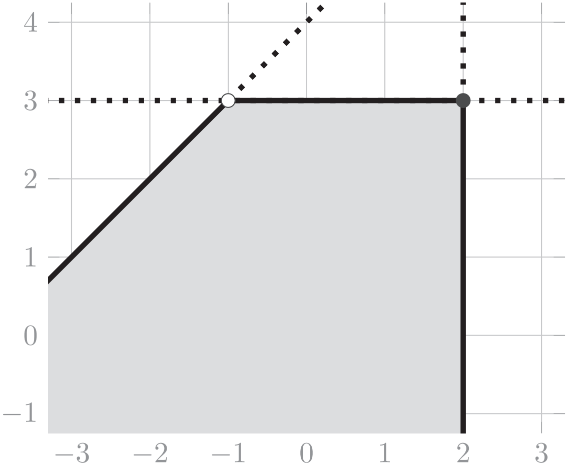
$$\begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix} \longrightarrow$$

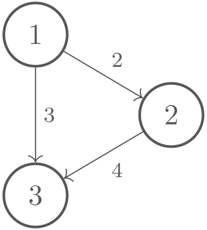


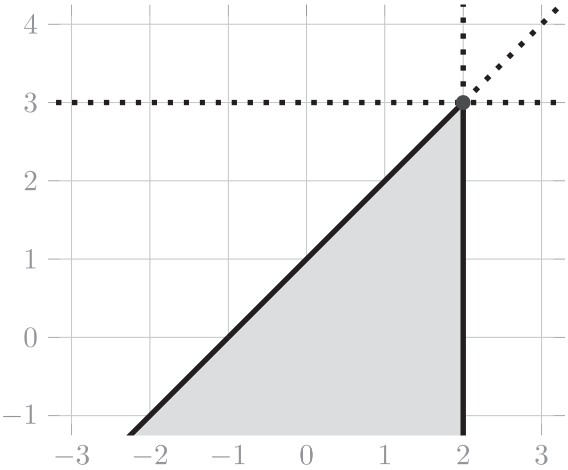


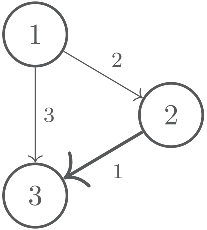


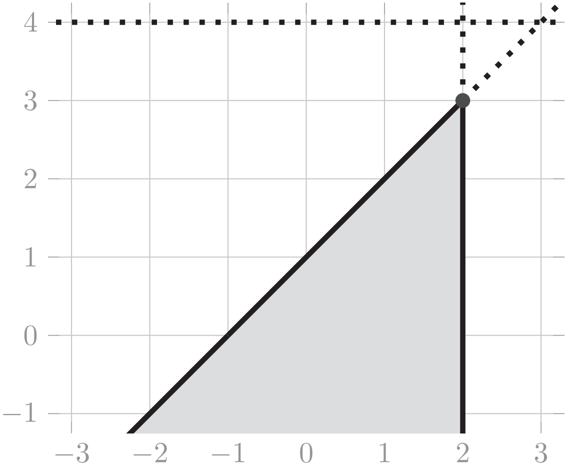
$$X = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix} \odot Z$$

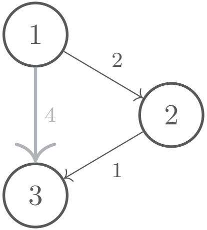

4
3
2
1
0
−1
−3
−2
−1
0
1
2
3

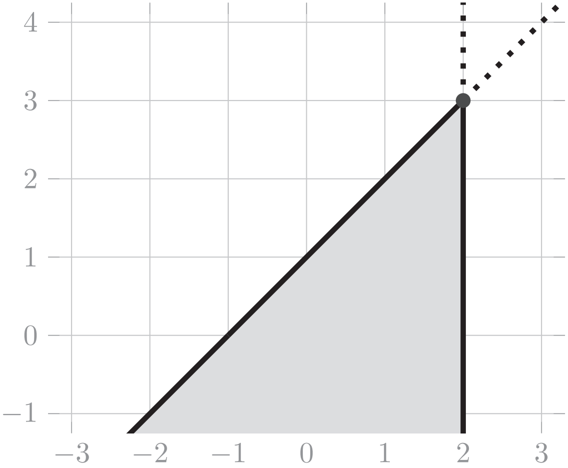

1
2
3
2
3
4

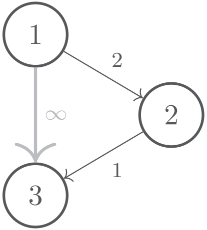

4
3
2
1
0
−1
−3
−2
−1
0
1
2
3


1
2
3
2
3
1


4
3
2
1
0
−1
−3
−2
−1
0
1
2
3


1
2
3
2
4
1


4
3
2
1
0
−1
−3
−2
−1
0
1
2
3


1
2
3
2
1
∞

Covering relations

Definition

For a given digraph G, the *transitive reduction* G^t is the digraph containing an edge $j \to i$ whenever $j \to i$ is an edge in G and there is no other path between j and i.

Covering relations

Definition

For a given digraph G, the *transitive reduction* G^t is the digraph containing an edge $j \to i$ whenever $j \to i$ is an edge in G and there is no other path between j and i.

The set of edges E^t of G^t are called the *covering relations* of G.

Weighted transitive reduction

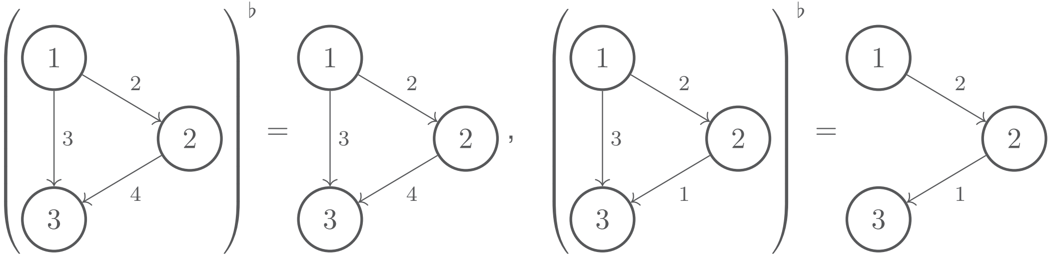
Definition

For a DAG G with weighted adjacency matrix $C \in \mathbb{T}^{n \times n}$ define the *weighted transitive reduction* $G^{\flat}_{C}$ with weight matrix $C^{\flat} \in \mathbb{T}^{n \times n}$ as the graph containing an edge $j \to i$ if and only if this edge is the unique shortest path connecting j with i.

Weighted transitive reduction

Definition

For a DAG G with weighted adjacency matrix $C \in \mathbb{T}^{n \times n}$ define the *weighted transitive reduction* $G_C^\flat$ with weight matrix $C^\flat \in \mathbb{T}^{n \times n}$ as the graph containing an edge $j \to i$ if and only if this edge is the unique shortest path connecting j with i.



Weighted transitive reduction

Definition

For a DAG G with weighted adjacency matrix $C \in \mathbb{T}^{n \times n}$ define the *weighted transitive reduction* $G_C^\flat$ with weight matrix $C^\flat \in \mathbb{T}^{n \times n}$ as the graph containing an edge $j \to i$ if and only if this edge is the unique shortest path connecting j with i.

Proposition

For every $C \in \mathbb{T}^{n \times n}$ lower-triangular, we have $G^t \subseteq G_C^\flat \subseteq G$.

Minimal facet description

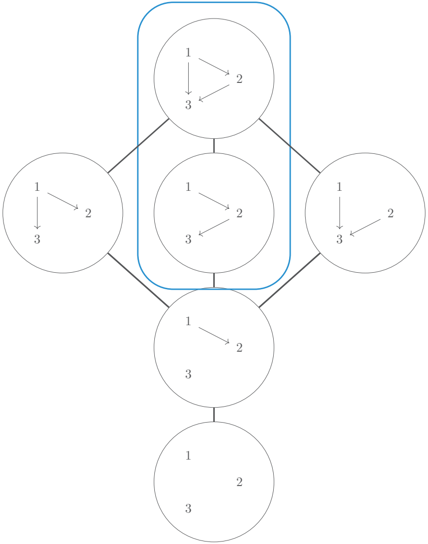
Theorem (Améndola and 🚢 2024)

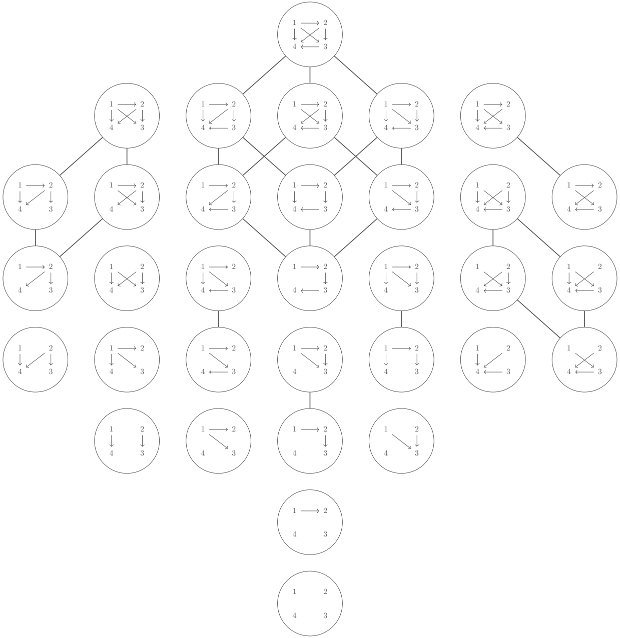
For $C \in \mathbb{T}^{n \times n}$ *lower-triangular, the set of matrices* $C' \in \mathbb{T}^{n \times n}$ *such that* $\mathrm{Q}(C) = \mathrm{Q}(C')$ *is the polyhedral cone*

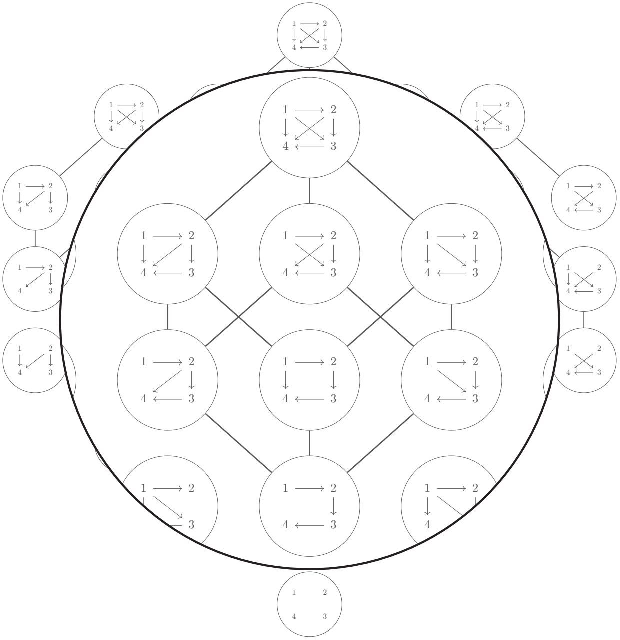
$$C^\star + \mathrm{pos}(e_{ij} \mid j \to i \in G^\star \setminus G^\flat).$$

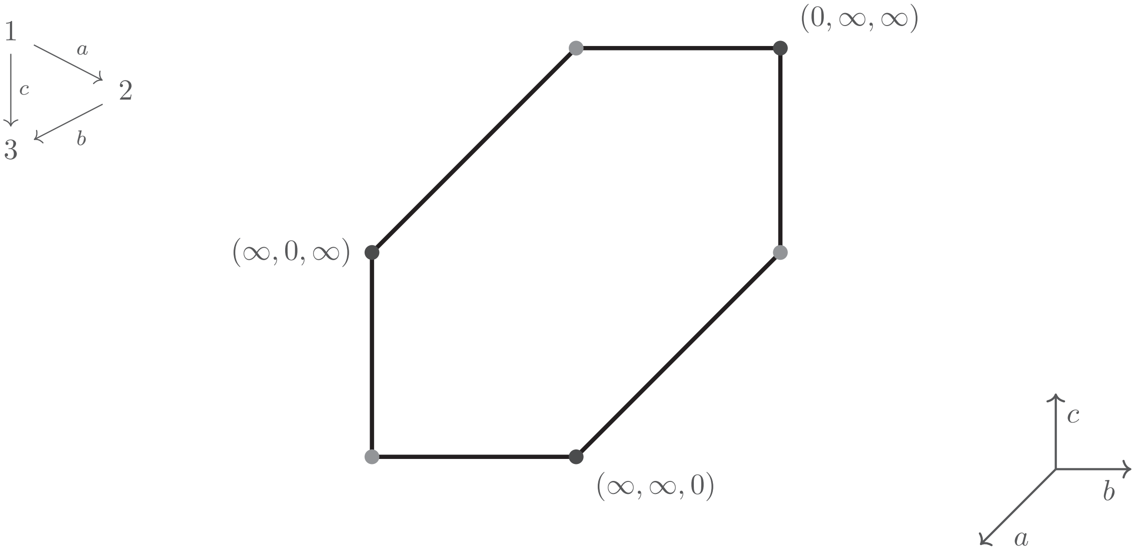
Theorem (Améndola and 🚢 2024)

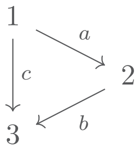
Let $C \in \mathbb{T}^{n \times n}$ *be the weighted adjacency matrix of a DAG* G. *Then,* C *gives a minimal facet description of* $\mathrm{Q}(C)$ *if and only if* $G = G^\flat_C$.

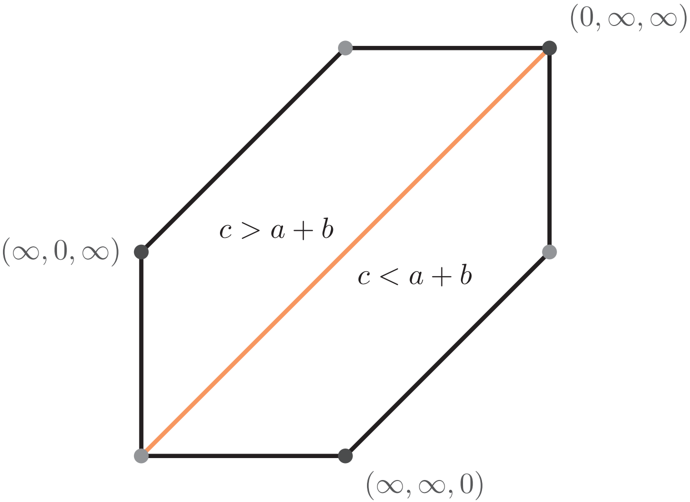

1
2
3
1
2
3
1
2
3
1
2
3
1
2
3
1
2
3

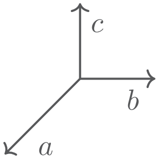


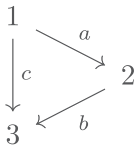


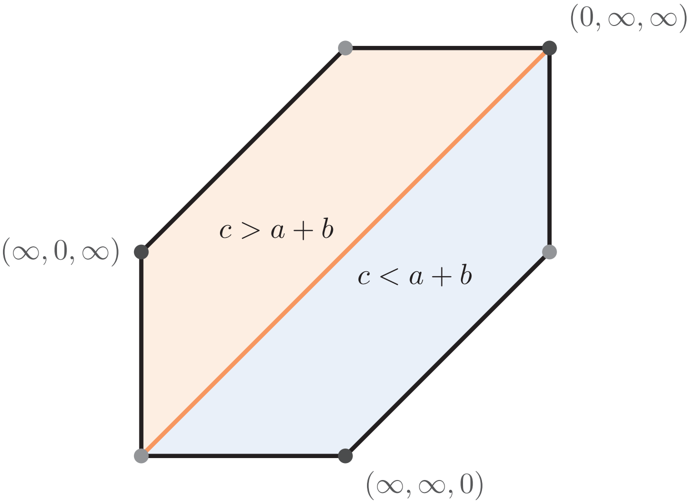

1
a
c
2
b
3
(0, ∞, ∞)
(∞, 0, ∞)
(∞, ∞, 0)
c
b
a

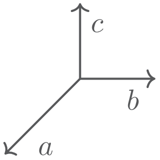

1
2
3
a
b
c

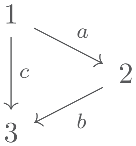

$(0, \infty, \infty)$
$(\infty, 0, \infty)$
$(\infty, \infty, 0)$
$c > a + b$
$c < a + b$

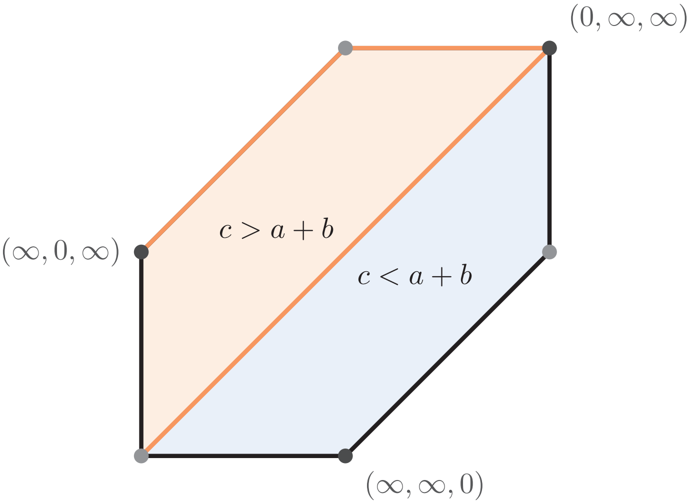

a
b
c

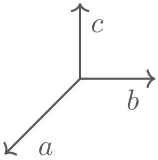

1
2
3
a
b
c

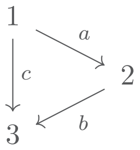

(0, ∞, ∞)
(∞, 0, ∞)
(∞, ∞, 0)
c > a + b
c < a + b

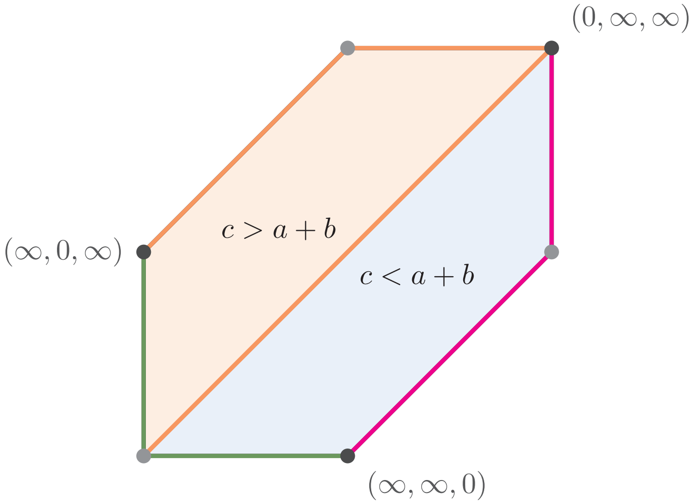

c
b
a

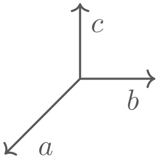

1
a
c
2
3
b

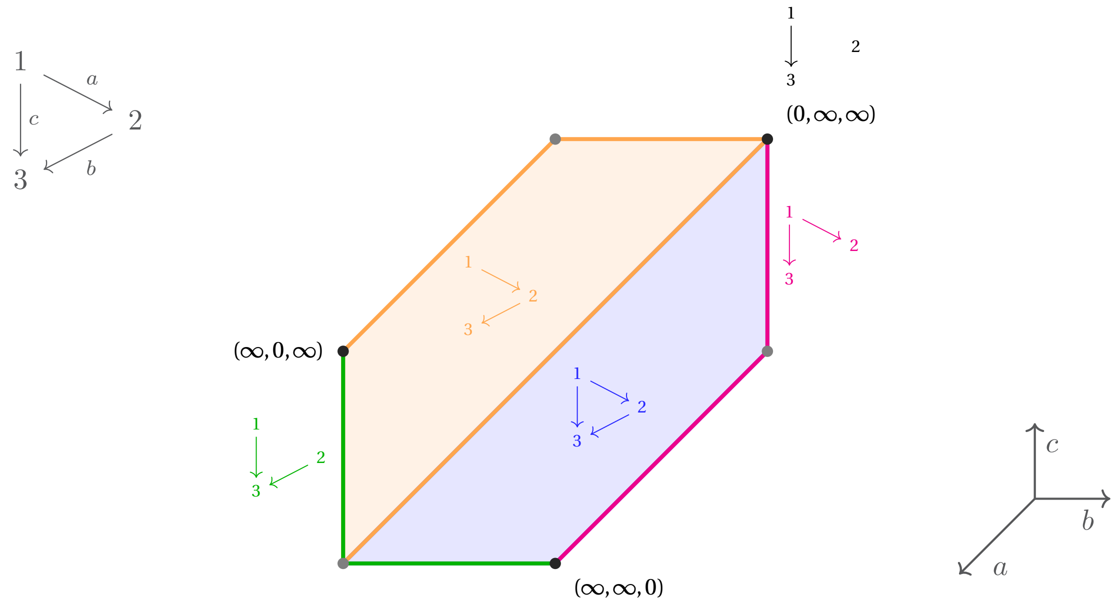

$(0, \infty, \infty)$
$(\infty, 0, \infty)$
$c > a + b$
$c < a + b$
$(\infty, \infty, 0)$


c
b
a


1
a
c
2
b
3


$(0, \infty, \infty)$
$c > a + b$
$(\infty, 0, \infty)$
$c < a + b$
$(\infty, \infty, 0)$


c
b
a


1
a
c
2
3
b
1
2
3
(0,∞,∞)
(∞,0,∞)
(∞,∞,0)
c
b
a

Bibliography I

- Améndola, Carlos and  (2024). *Tropical combinatorics of max-linear Bayesian networks*. arXiv: 2411.10394 [math.CO] (cit. on p. 30).
- Butkovič, Peter (2010). *Max-linear Systems: Theory and Algorithms*. Springer Monographs in Mathematics. London: Springer. DOI: 10.1007/978-1-84996-299-5 (cit. on p. 15).
- Joswig, Michael and Katja Kulas (2010). "Tropical and ordinary convexity combined". In: *Advances in Geometry* 10.2, pp. 333–352. ISSN: 1615-7168. DOI: 10.1515/advgeom.2010.012 (cit. on p. 14).
- Joswig, Michael and Georg Loho (2016). "Weighted digraphs and tropical cones". In: *Linear Algebra and its Applications* 501, pp. 304–343. ISSN: 00243795. DOI: 10.1016/j.laa.2016.02.027 (cit. on p. 15).
- Puente, María Jesús de la (2013). "On tropical Kleene star matrices and alcoved polytopes". In: *Kybernetika* 49.6, pp. 897–910. ISSN: 0023-5954. DOI: 10338.dmlcz/143578 (cit. on p. 15).