Tropical combinatorics of weighted DAG polytropes

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Gröbner fans and toric ideals

Let $\mathcal{D}$ be a digraph with edge set E. The *all-pairs shortest-path ideal* I_s of $\mathcal{D}$ is the toric ideal

$$I_s = \langle e_{ij}e_{ji} - 1, e_{ij} - e_{ik}e_{kj} \rangle \cap \mathbb{K}[E]$$

for $1 \le i,j,k \le n$ distinct and $\mathbb{K}[E] = \mathbb{K}[e_{ij} \mid j \to i \in E]$ is the *edge ring* of a digraph $\mathcal{D}$. Each $C \in \mathbb{T}^{n\times n}$ gives weights on the variables. This gives us the initial ideal $\mathrm{in}_C(I_s)$.

Fine classification

Two polytropes $P = \mathrm{Q}(C)$ and $P' = \mathrm{Q}(C')$ are tropically equivalent if and only if

$$\mathrm{in}_C(I_s) = \mathrm{in}_{C'}(I_s).$$

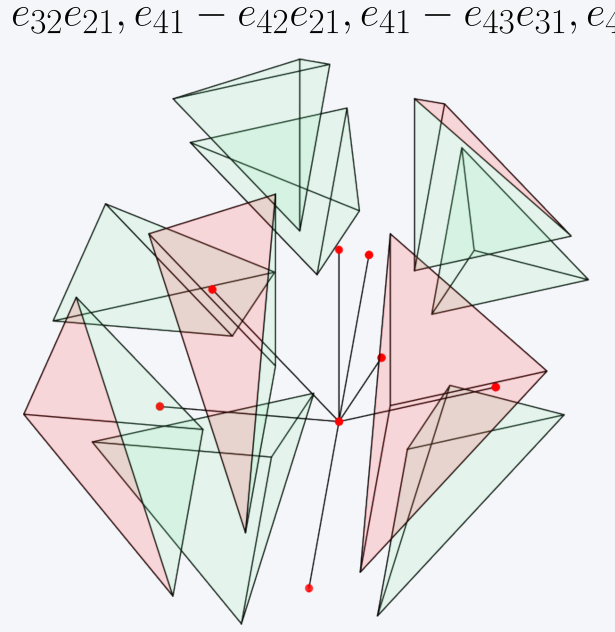
The case $\mathcal{D} = K_n$ is treated in [4].

Counting Gröbner cones up to tropical equivalence

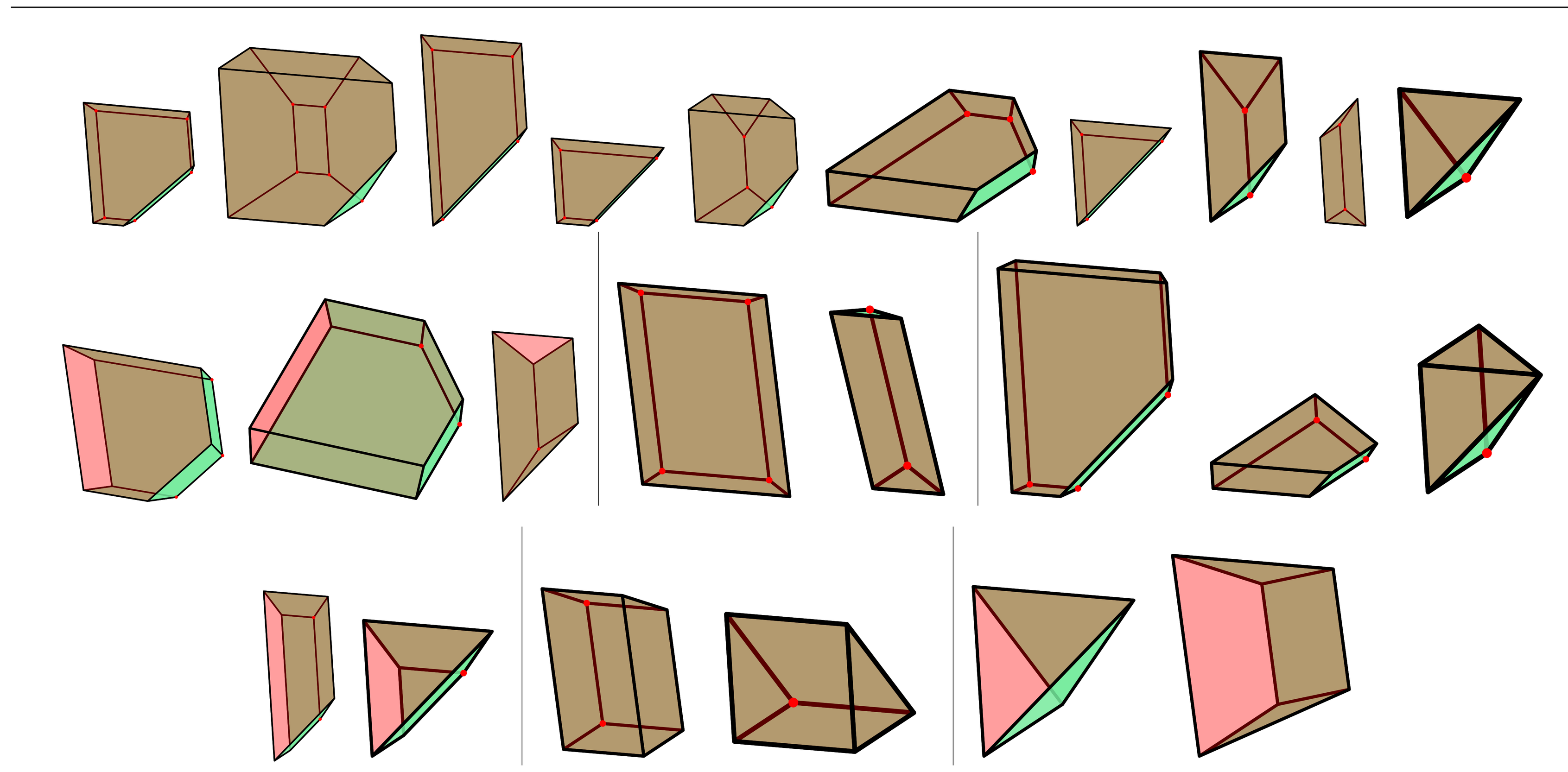
n	1	2	3	4	5	6	7
	1	2	6	33	≥400	≥16251	≥2717966

Example: Gröbner fan for the complete DAG $\mathcal{D}$ on four nodes

$$I_s = \langle e_{31} - e_{32}e_{21}, e_{41} - e_{42}e_{21}, e_{41} - e_{43}e_{31}, e_{42} - e_{43}e_{32} \rangle$$



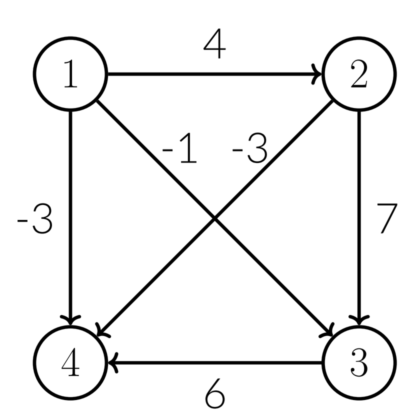
Case study: combinatorial types for $n = 4$ with ≥ 2 facets

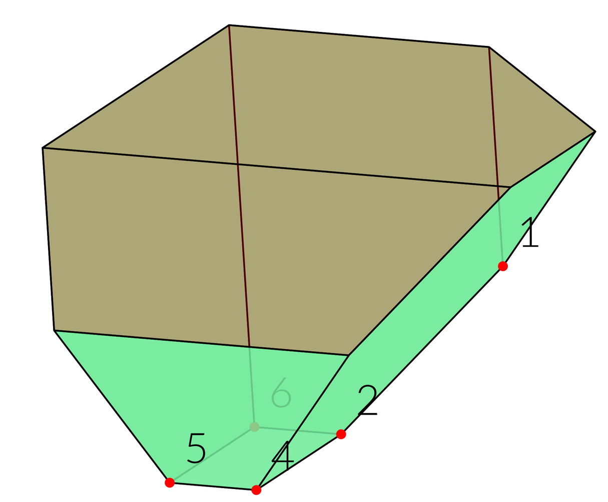


What are polytropes?

A *polytrope* is a tropical polyhedron that is also classically convex.

A weighted digraph and its polytrope





Tropical convex hull and weighted digraph polyhedron

$$\mathbb{T} := \mathbb{R} \cup \{\infty\}, \quad x \oplus y := \min\{x,y\}, \quad x \odot y = x + y$$
$$\mathbb{TP}^{n-1} := \mathbb{T}^n \setminus \{(\infty,\infty,\dots,\infty)\}/\mathbb{R}(1,1,\dots,1)$$
$$\mathrm{tconv}(V) := \left\{ \lambda_1 \odot v^{(1)} \oplus \dots \oplus \lambda_n \odot v^{(n)} \;\middle|\; \lambda_i \in \mathbb{T}, v^{(j)} \in V \right\}, \quad V \subset \mathbb{TP}^{n-1}$$
$$\mathrm{Q}(C) := \left\{ x \in \mathbb{TP}^{n-1} \mid x_i - x_j \le c_{ij} \text{ for all } 1 \le i,j \le n, i \ne j \right\}, \quad C \in \mathbb{T}^{n\times n}$$

The Kleene star

For $C \in \mathbb{T}^{n\times n}$, its *Kleene star* is defined as $C^\star = I \oplus C \oplus C^{\odot 2} \oplus \dots \oplus C^{\odot(n-1)}$.

Relationship between tropical convex hull and weighted digraph polyhedron

A set $P \subset \mathbb{TP}^{n-1}$ is a polytrope if and only if $P = \mathrm{tconv}(C^\star) = \mathrm{Q}(C) = \mathrm{Q}(C^\star)$ for some $C \in \mathbb{T}^{n\times n}$.

What is tropical combinatorial equivalence?

Tropical hyperplanes

For $v \in \mathbb{TP}^{n-1}$, a max-tropical hyperplane the affine part of $\mathbb{TP}^{n-1}$ is given by a max-tropical polynomial

$$\alpha_v := -v_1 \boxdot x_1 \boxplus -v_2 \boxdot x_2 \boxplus \dots \boxplus -v_d \boxdot x_d$$

where $x \boxplus y = \max\{x,y\}$ and $x \boxdot y = x + y$.

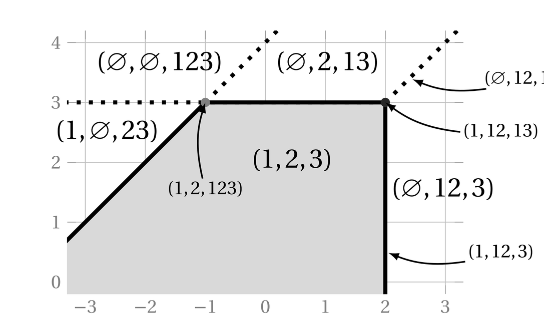


Figure (1). Tropical polyhedron with hyperplane arrangement

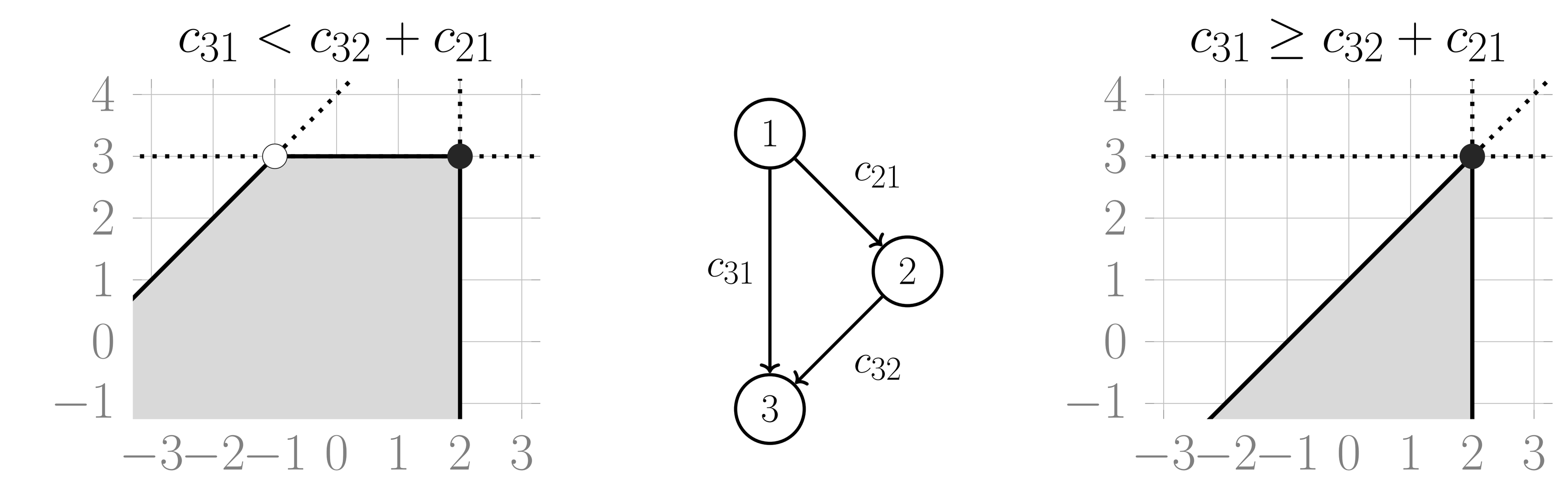
Tropical hyperplane arrangements and covectors

Any tropical polyhedron has a surrounding tropical hyperplane arrangement.
A tropical covector tracks the terms in α_v that realize the maximum at any given point.

Tropical combinatorial equivalence

Two point configurations $V = (v_1,\dots,v_n)$ and $W = (w_1,\dots,w_n)$ of points in $\mathbb{TP}^{d-1}$ are *tropically equivalent* if there is a permutation of the vertices and the coordinates inducing an isomorphism on the poset of tropical covectors.

Non-uniqueness of facet descriptions



The transitive reduction to our rescue

For a given digraph $\mathcal{D}$, the *transitive reduction* $\mathcal{D}^t$ contains an edge $j \to i$ iff $j \to i$ is an edge in $\mathcal{D}$ and there is no other path between j and i.

For the *weighted transitive reduction* $\mathcal{D}^\flat$, we additionally demand that an edge $j \to i$ with weight c_{ij} is the unique shortest path between j and i in $\mathcal{D}$.

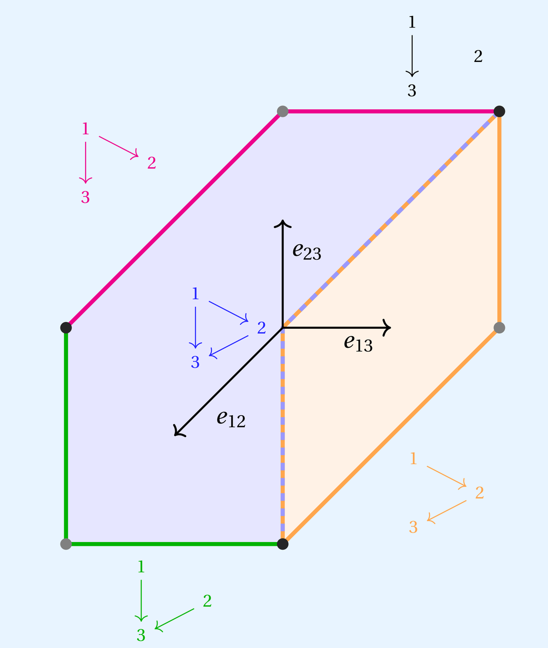
Coarse classification

How much can we vary facets without changing $\mathrm{Q}(C)$?
For $C \in \mathbb{T}^{n\times n}$ lower-triangular, the set of matrices $C' \in \mathbb{T}^{n\times n}$ such that $\mathrm{Q}(C) = \mathrm{Q}(C')$ is the polyhedral cone

$$C^\star + \mathrm{pos}(e_{ij} \mid j \to i \in \mathcal{D}^\star \setminus \mathcal{D}^\flat).$$

What is the unique minimal facet description?
Let $C \in \mathbb{T}^{n\times n}$ be lower-triangular. Then, $C^\flat$ is the unique matrix with minimal support such that $\mathrm{Q}(C)$ and $\mathrm{Q}(C^\flat)$ coincide.

Can we find the transitive reduction from the algebra?
The edges of $\mathcal{D}^\flat$ correspond to $e_{ij} \notin \mathrm{in}_C\, I_{s,\mathcal{D}}$.



References

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