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Waving the flag for max-linear Bayesian networks

(the sequel to `arXiv:2411.10394` by Améndola and 🛳)

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Tropical numbers

Definition

The *min-plus tropical semiring* is $\mathbb{T} := \mathbb{T}_{\min} := (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ where $\oplus := \min$ and $\odot := +$.

Alternatively, $\mathbb{T}_{\max} := (\mathbb{R} \cup \{-\infty\}, \max, +)$, $(\mathbb{R}_{\geq 0}, \min, \cdot)$, and $(\mathbb{R}_{\geq 0}, \max, \cdot)$ are equally valid choices for tropical semirings.

$$\begin{array}{ccc} \mathbb{T}_{\min} & \xleftrightarrow{\;\cdot(-1)\;} & \mathbb{T}_{\max} \\ \log \uparrow \;\; \downarrow \mathit{exp} & & \log \uparrow \;\; \downarrow \mathit{exp} \\ (\mathbb{R}_{\geq 0}, \min) & \xleftrightarrow{\;\frac{1}{\cdot}\;} & (\mathbb{R}_{\geq 0}, \max) \end{array}$$

Max-linear Bayesian networks

Definition (Gissibl and Klüppelberg 2018)

A max-linear Bayesian network is a random vector $X = (X_1, \dots, X_N)$ where the random variables X_i can be recursively expressed as

$$X_i = \bigvee_{j=1}^{n} c_{ij} X_j \vee Z_i.$$

The expression for the X_i can be interpreted as the recursive affine equation

$$X = C \cdot X \vee Z.$$

Networks and shortest-paths

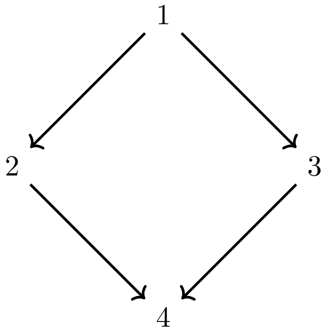
he expression for the X_i can be interpreted as the recursive affine equation

$$X = C \cdot X \vee Z.$$

Since $C^k = C^{k+1}$ for $k \gg 0$, we can solve this recursive equation to get the expression

$$X = \underbrace{\left(I_n \vee C \vee \cdots \vee C^{n-1}\right)}_{=:C^\star} \cdot Z.$$

The matrix $C^\star$ is called the *Kleene star*.



$$X_1 = Z_1$$
$$X_2 = c_{21}X_1 \vee Z_2$$
$$X_3 = c_{31}X_3 \vee Z_3$$
$$X_4 = c_{42}X_2 \vee c_{43}X_3 \vee Z_4$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 1 \\ c_{21} \\ c_{31} \\ 0 \end{pmatrix} \cdot X_1 \vee \begin{pmatrix} 0 \\ 1 \\ 0 \\ c_{42} \end{pmatrix} \cdot X_2 \vee \begin{pmatrix} 0 \\ 0 \\ 1 \\ c_{43} \end{pmatrix} \cdot X_3 \vee \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot X_4 \vee \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 1 \\ c_{21} \\ c_{31} \\ c_{42}c_{21} \vee c_{43}c_{31} \end{pmatrix} \cdot Z_1 \vee \begin{pmatrix} 0 \\ 1 \\ 0 \\ c_{42} \end{pmatrix} \cdot Z_2 \vee \begin{pmatrix} 0 \\ 0 \\ 1 \\ c_{43} \end{pmatrix} \cdot Z_3 \vee \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot Z_4$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_{21} & 1 & 0 & 0 \\ c_{31} & 0 & 1 & 0 \\ c_{42}c_{21} \vee c_{43}c_{31} & c_{42} & c_{43} & 1 \end{pmatrix} \cdot \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}$$

Tropical polytopes and polytropes

Definition

The *tropical convex hull* of $V \subset \mathbb{TP}^{d-1}$ finite is the *(min-plus)-linear span* of V, i. e.

$$\operatorname{tconv}(V) := \left\{ \lambda_1 \odot v^{(1)} \oplus \cdots \oplus \lambda_n \odot v^{(n)} \;\middle|\; \lambda_i \in \mathbb{T}, v^{(j)} \in V \right\}.$$

Note: A tropical polytope P is the column span of a matrix with the vertices of P as columns.

Definition (Joswig and Kulas 2010)

A tropical polytope P is called a polytrope if it is classically convex.

Polytropes and Kleene stars

Theorem (Joswig and Loho 2016)

A tropical polytope $P \subset \mathbb{TA}^{d-1}$ *is a polytrope if and only if*

$$P = \mathrm{tconv}(C^\star) = \mathrm{Q}(C)$$

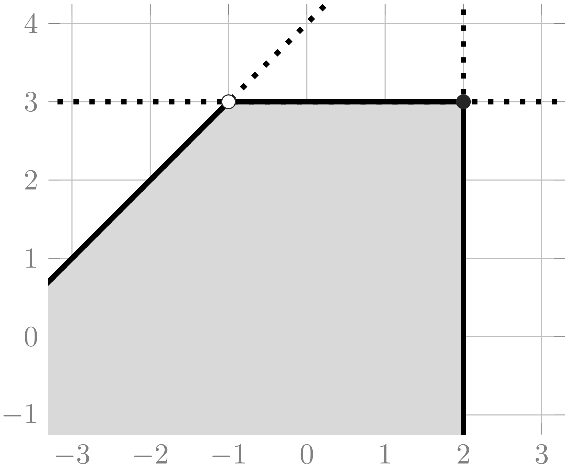
for some $C \in \mathbb{T}^{n \times n}$.

Definition

The *weighted digraph polyhedron* $\mathrm{Q}(C)$ of a matrix $C \in \mathbb{T}^{n \times n}$ is given by

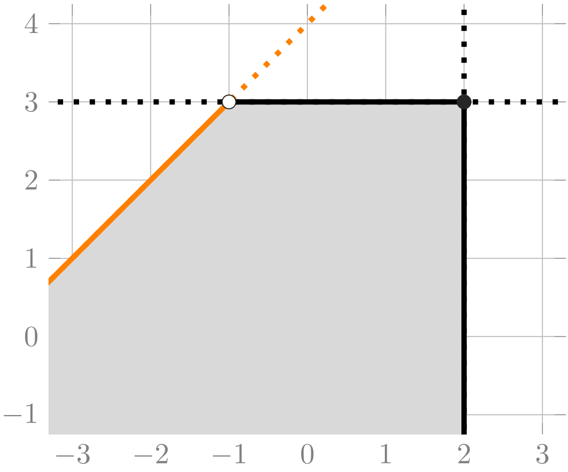
$$\mathrm{Q}(C) = \{\, x \in \mathbb{TP}^{n-1} \mid x_i - x_j \leq c_{ij} \text{ for all } 1 \leq i, j \leq n, i \neq j \,\}.$$

An actual living polytrope



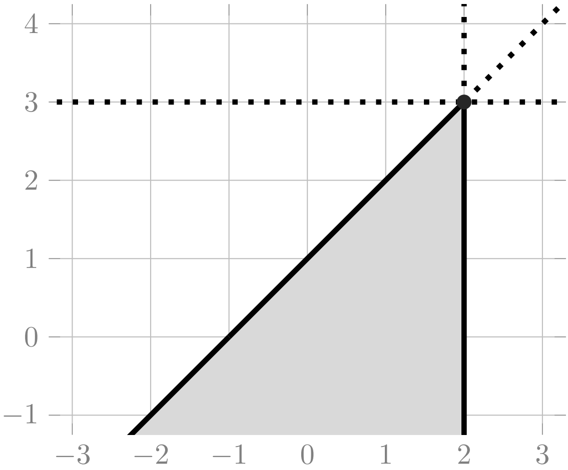
$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 4 & 0 \end{pmatrix}$$

Getting into the details



$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & \color{orange}{4} & 0 \end{pmatrix}$$

Getting into the details



$$C = \begin{pmatrix} 0 & \infty & \infty \\ 2 & 0 & \infty \\ 3 & 1 & 0 \end{pmatrix}$$

$\rightsquigarrow c_{31} \oplus c_{32} c_{21}$

Linear spaces and how to parametrize them

Any n-dimensional linear space $V \subseteq k^d$ can be represented as the column span of a $d \times n$-matrix C. The variety parametrizing those linear spaces is called the *Grassmannian* $\mathrm{Gr}(n, d)$.

Definition

The rational map $\pi \colon k^{d \times n} \dashrightarrow \mathrm{Gr}(n, d)$ sending a $d \times n$-matrix C to its $d \times d$-minors is called the *Stiefel map*.

Flags of linear spaces and how to parametrize them

Definition

A *complete flag* of linear spaces in k^d is a sequence of linear spaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_d = k^d$$

with $\dim(V_i) = i$.

A complete flag can also be represented by a $d \times d$-matrix. In particular, the first column needs to span V_1, the first two V_2, and so on.

The variety parametrizing complete flags is called the *complete flag variety* Fl_d.

Enter the flag Stiefel map

A complete flag can also be represented by a $d \times d$-matrix. In particular, the first column needs to span V_1, the first two V_2, and so on.

Definition

The rational map $\pi_{\text{flag}} \colon k^{d \times d} \dashrightarrow \mathrm{Fl}_d$ sending a $d \times d$-matrix C to the 1×1-minors of the first column, the 2×2-minors of the first two columns, and so on is called the *flag Stiefel map*.

Remark

This means the complete flag variety Fl_d *embeds into a product of Grassmannians*

$$\mathrm{Gr}(1, d) \times \mathrm{Gr}(2, d) \times \cdots \times \mathrm{Gr}(d-1, d).$$

Tropicalizing the Stiefel map

Tropicalizing the Stiefel map gives a map assigning to a tropical matrix the vector of tropical maximal minors, and we can do the same with the flag Stiefel map.

For the previous example, we get the following map

$$\begin{pmatrix} 0 & \infty & \infty \\ c_{21} & 0 & \infty \\ c_{31} & c_{32} & 0 \end{pmatrix} \mapsto (0, c_{21}, c_{31}) \times (0, c_{32}, c_{32}c_{21} \oplus c_{31}).$$

A bigger picture

For a DAG on $n = 4$ nodes, we look at a matrix with Kleene star

$$C^\star = \begin{pmatrix} 0 & \infty & \infty & \infty \\ c_{21} & 0 & \infty & \infty \\ c_{31} \oplus c_{32}c_{21} & c_{32} & 0 & \infty \\ c_{41} \oplus c_{43}c_{31} \oplus c_{42}c_{21} \oplus c_{43}c_{32}c_{21} & c_{42} \oplus c_{43}c_{32} & c_{43} & 0 \end{pmatrix}.$$

The matrix C itself gets mapped by the flag Stiefel map to

$$\begin{aligned} &(0, c_{21}, c_{31}, c_{41}) \\ \times&(0, c_{32}, c_{42}, c_{31} \oplus c_{32}c_{21}, c_{41} \oplus c_{42}c_{21}, c_{31}c_{42} \oplus c_{41}c_{32}) \\ \times&(0, c_{43}, c_{42} \oplus c_{43}c_{32}, c_{41} \oplus c_{43}c_{31} \oplus c_{42}c_{21} \oplus c_{43}c_{32}c_{21}). \end{aligned}$$

Theorem

A tropical polytope $P \subset \mathbb{TA}^{d-1}$ *is a polytrope if and only if*

$$P = \mathrm{tconv}(C^\star) = \mathrm{Q}(C)$$

for some $C \in \mathbb{T}^{n \times n}$.

Conjecture

The full-dimensional polytropes in $\mathbb{TA}^{d-1}$ *associated to DAGs are parametrized by ...*

1. *... a tropical prevariety inside* $\mathrm{trop}(\mathrm{Fl}_d)$.
2. *... a tropical subvariety of* $\mathrm{trop}(\mathrm{Fl}_d)$.

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