Berlin Mathematics Research Center MATH+

Tropical Combinatorics of Max-Linear Bayesian Networks

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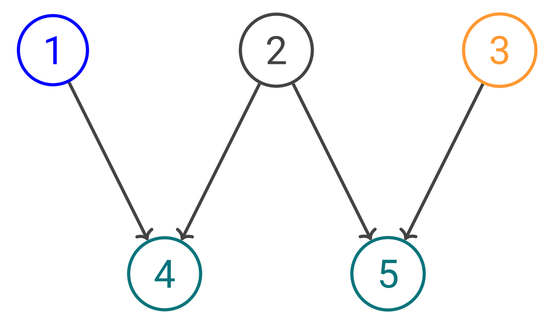
Bayesian networks and graphical models

Max-linear Bayesian networks have emerged from the need to model cause and effect relations between large observed values of several variables. They are a type of graphical model, where the causal relations between the random variables can be modeled by a **di**rected **graph** (digraph).
For the case of max-linear Bayesian networks, these relations take the form of structural equations over the *max-times tropical semi-ring*. This means every random variable X_i is the weighted maximum over the other random variables X_j and a independent random variable Z_i.



$$X_1 = Z_1$$
$$X_2 = Z_2$$
$$X_3 = Z_3$$
$$X_4 = c_{41}X_1 \vee c_{42}X_2 \vee Z_4$$
$$X_5 = c_{52}X_2 \vee c_{53}X_3 \vee Z_5$$

Figure 1. The underlying digraph of a max-linear Bayesian network and the associated structural equations

We can express the structural equations purely in terms of the Z_i, which gives

$$X_i = \bigvee_{j=1}^{n} c_{ij}^{\circledast} Z_j \tag{1}$$

where $c_{ij}^{\circledast}$ denote the entries of the *max-times Kleene star* $C^{\circledast} := I_n \vee C \vee C^{\circledast 2} \vee \cdots \vee C^{\circledast n}$ where the expression is evaluated in the max-times semi-ring.
But this means in particular that $X = C^{\circledast}Z$ and the support of the random vector $X = (X_1, \ldots, X_n)$ is the image of the tropical linear map given by matrix multiplication with $C^{\circledast}$. At the same time, we can apply $-\log$ as a transformation to get an object in *tropical projective space* $\mathbb{TP}^{n-1}$ over the *min-plus tropical semi-ring*, which is useful for questions of convexity.

Polytropes

A *polytope* is the convex hull of finitely many points, or equivalently, the bounded intersection of finitely many closed half-spaces. It is one of the main objects of interest in *discrete geometry* and a convex set.
Likewise, the same definitions can be tropicalized by considering all definitions in tropical arithmetic. This gives us notions of tropical convexity and tropical polytopes. It happens to be that a tropical polytope is exactly the image of matrix multiplication with the matrix having the tropical vertices as rows [2, Proposition 4]. In other words, tropical polytopes are exactly the images of tropical linear maps as is the support of a max-linear Bayesian network.
Polytopes that happen to be both ordinarily and tropically convex are special and have been dubbed *polytropes* by Joswig and Kulas [3]. For example, from Develin and Sturmfels [2] we know that there are 35 combinatorial types of tropical quadrilaterals, but only two out of them are polytropes.

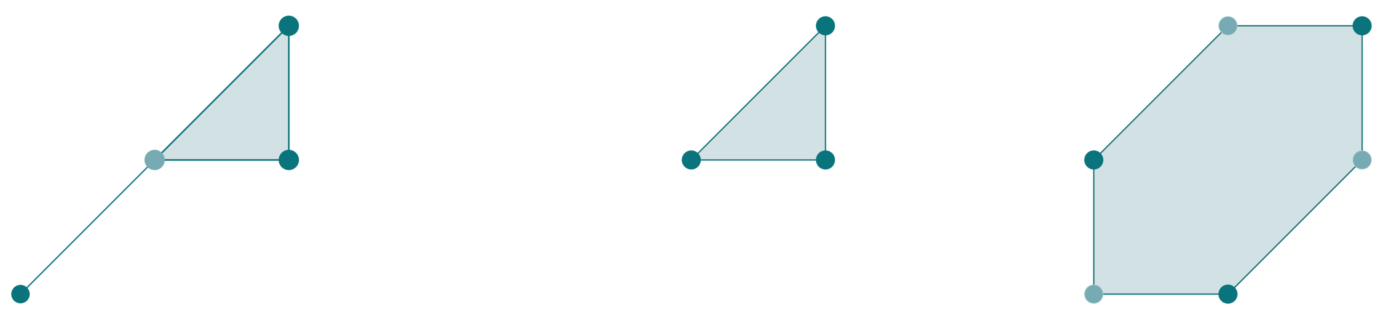


Figure 2. Three types of tropical 2-simplices, of which only the last two are polytropes.

Polytropes from weighted digraphs

The underlying graph of a max-linear Bayesian networks together with the coefficients c_{ij} gives a *weighted digraph*. As proposed by Joswig and Loho [4], we can construct a polyhedron from that which is conveniently named the *weighted digraph polyhedron* $\mathrm{Q}(A)$. It is the intersection of the half-spaces that satisfy the inequalities

$$x_i - x_j \leq c_{ij}$$

for a matrix $C \in \mathbb{R}^{n\times n}$. Thus, the weighted digraph polyhedron is a classically convex set.
On the other hand, a weighted digraph polyhedron can also be written as the intersection of closed *tropical half-spaces*, which can be thought of all points $x \in \mathbb{TP}^{n-1}$ satisfying an inequality of the form

$$c' \odot x \leq c'' \odot x \tag{2}$$

for points $c', c'' \in \mathbb{TP}^{n-1}$ without common finite coordinates. These tropical half-spaces are indeed tropically convex, which makes a weighted digraph polyhedron tropically convex as well.
We can get such tropical half-spaces by choosing a point $a \in \mathbb{TP}^{n-1}$ and choosing an index $i \in \{1, \ldots, n\}$. Then, the tropical hyperplane $\mathcal{T}$ with apex $-a$ is given by the vanishing locus of

$$\mathcal{T} = a_1 \odot x_1 \oplus \cdots \oplus a_n \odot x_n$$

and divides $\mathbb{TP}^{n-1}$ into n sectors. The *i-th closed sector* $\mathrm{S}_i(a)$ is the set satisfying the inequality

$$a_i \odot x_i \leq \min_{j \neq i} a_j \odot x_j,$$

or in the language of eq. (2), the tropical half-space obtained by setting $c' = (\infty, \ldots, a_i, \ldots, \infty)$ and $c'' = (a_1, \ldots, a_{i-1}, \infty, a_{i+1}, \ldots, a_n)$. With this language, we can link the tropical polytope and the weighted digraph polyhedron of a Kleene star matrix:

Theorem
Let $C \in \overline{\mathbb{R}}^{n\times n}$ be the weight matrix of a DAG and $A = C^*$. Then, in $\mathbb{TA}^{n-1}$ we have

$$\mathrm{tconv}(A) = \mathrm{Q}(A) = \bigcap_{i=1}^{n} \mathrm{S}_i(a_i) = -\left(\bigcap_{j=1}^{n} \mathrm{S}_j(a^{(j)})\right).$$

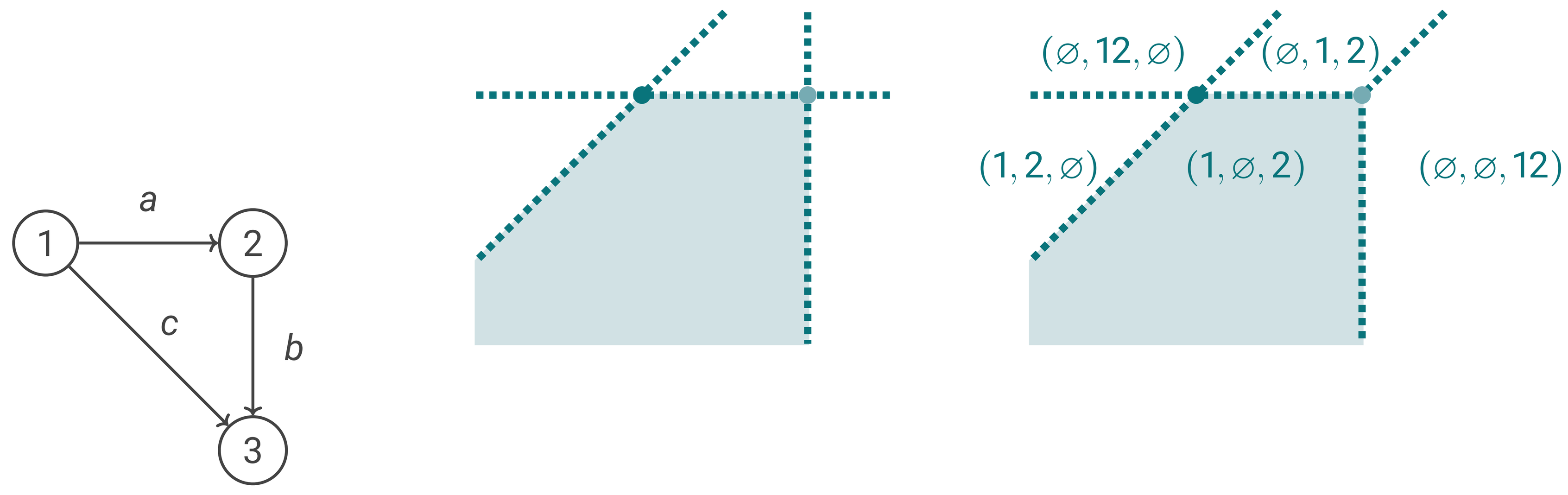


Figure 3. The weighted digraph polyhedron $\mathrm{Q}(A)$ for a digraph on three nodes with matrix $A = \begin{pmatrix} 0 & \infty & \infty \\ a & 0 & \infty \\ c & b & 0 \end{pmatrix}$,the supporting (tropical) half-spaces and the types of the associated subdivision.

Covector decomposition of tropical projective space

The set of vertices of $\mathrm{tconv}(A)$ also induces a subdivision of the ambient tropical projective space via the tropical hyperplanes associated to each vertex.
For this, we define the *type of a point* $x \in \mathbb{TP}^{n-1}$ *w. r. t.* A as an ordered tuple $(S_1, \ldots, S_n)$ consisting of subsets $S_i \subset \{1, \ldots, n\}$ defined as follows: An index i is in S_j if

$$a_{ij} - x_j = a_{i1} - x_1 \oplus \cdots \oplus a_{in} - x_n.$$

Expressing this with sectors, a type describes the intersection of sectors and every set S_j contains all the (indices of) vertices v_i, whose j-th sector is intersected with.
Now, we get a *covector decomposition* of $\mathbb{TP}^{n-1}$ as the normal complex of the union of tropical hyperplanes to each vertex. This happens to be dual to the subdivision of a product of simplices, which is a triangulation if and only if A is tropically regular.
Even if A is not tropically regular in our case, it is still interesting to ask how the singular maximal minors affect the structure of $\mathrm{tconv}(A)$.

Separation in graphs and conditional independence

One can also derive conditional independence statements $I \perp\!\!\!\perp J \mid K$ (for disjoint index sets $I, J, K \subset [n]$) from connectivity in the underlying digraph.
More specifically, two nodes i and j are *d-connected* given another set K if there exists a path between i and j such that

1. any head-to-head node occurring on the path is in K or an ancestor of K and
2. any other node is not in K.

Similarly, two nodes are $\star$-connected given another set K if there is a d-connecting path with at most one head-to-head node. Otherwise, i and j are d- resp. $\star$-separated.
For example, in Figure 1, the nodes 1 and 3 are $\star$-separated but not d-separated by the set $\{4, 5\}$ since there exists exactly one d-connecting path between 1 and 3, but it passes to head-to-head nodes. Compare this to the situation that $1 \perp\!\!\!\perp 3 \mid 4, 5$ in a max-linear Bayesian network, but not in a classical Gaussian network.
Now, analogously to the covariance matrix of a regular Gaussian Bayesian network, Améndola et al. defined the *tropical covariance matrix*

$$\Sigma^{\mathrm{trop}} = C^* \odot (C^*)^T.$$

In the case of Gaussian Bayesian networks it holds that $I \perp\!\!\!\perp J \mid K$ if and only if $\mathrm{rank}(\Sigma_{I\cup K, J\cup K}) = \#K$. Starting from this, a similar result was shown for max-linear Bayesian networks.

Theorem (Améndola, Hollering, Sullivant, Tran, [1])
Let G be a DAG and Σ^{trop} be supported on G. If K d-separates I and J in the DAG G then

$$\mathrm{trank}(\Sigma^{\mathrm{trop}}_{I\cup K, J\cup K}) = \#K$$

Outlook

- The tropical covariance matrix itself can be understood as a point configuration obtained by mapping the vertices of one polytrope onto another polytrope. An immediate question is how this point configuration behaves, if Σ^{trop} itself corresponds to a polytrope.
- A configuration of r points in $\mathbb{TP}^{n-1}$ defines a subdivision through the corresponding tropical hypersurfaces. This subdivision is dual to a subdivision of the product of simplices $\Delta_r \times \Delta_n$. This dual subdivision being a triangulation is equivalent to the non-vanishing of the minors of C^*. It is thus interesting to what extent the result due to Améndola et al. can be related to the rank and combinatorics of C^*.

References

[1] C. Améndola, B. Hollering, S. Sullivant, and N. Tran. Markov equivalence of max-linear bayesian networks. doi:10.48550/arXiv.2106.08305.

[2] M. Develin and B. Sturmfels. Tropical convexity. *Documenta Mathematica*, 9:1–27, 2004.

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[4] M. Joswig and G. Loho. Weighted digraphs and tropical cones. 501:304–343. doi:10.1016/j.laa.2016.02.027.